Permutations with Repetition

EXAMPLE 1 How many strings of length r can be formed from the English alphabet?
Solution: By the product rule, because there are 26 letters, and because each letter can be used repeatedly, we see that there are 26^r strings of length r.

THEOREM 1 The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combinations with Repetition

EXAMPLE 2 How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does **not** matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?
Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

The solution is the number of 4-combinations with repetition allowed from a three-element set.

EXAMPLE 3 How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.
Solution: Because the order in which the bills are selected does **not** matter and seven different types of bills can be selected as many as five times, this problem involves counting 5-combinations with repetition allowed from a set with seven elements. Listing all possibilities would be tedious, because there are a large number of solutions. Instead, we will illustrate the use of a technique for counting combinations with repetition allowed.

Suppose that a cash box has seven compartments, one to hold each type of bill, as illustrated in Figure 1. These compartments are separated by six dividers, as shown in the picture. The choice of five bills corresponds to placing five markers in the compartments holding different types of bills. Figure 2 illustrates this correspondence for three different ways to select five bills, where the six dividers are represented by bars and the five bills by stars.

The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars. Consequently, the number of ways to select the five bills is the number of ways to select the positions of the five stars, from 11 possible positions. This corresponds to the number of unordered selections of 5 objects from a set of 11 objects, which can be done in $C(11,5)$ ways. Consequently, there are

$$C(11,5) = \frac{11!}{5!\,6!} = 462$$

ways to choose five bills from the cash box with seven types of bills.

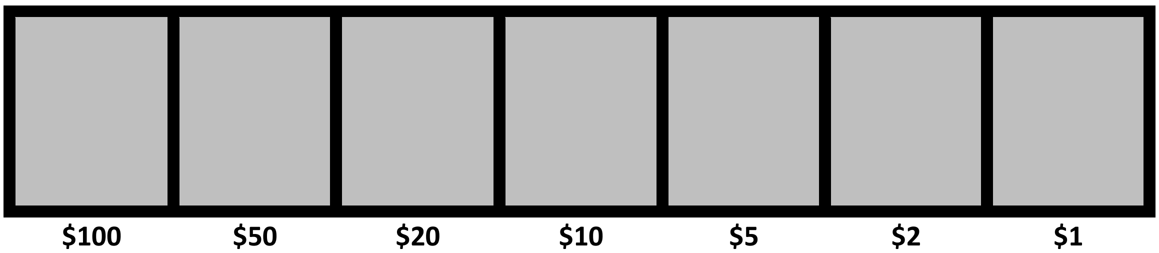

Figure 1. Cash Box with Seven Types of Bills

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Figure 2. Bars and Stars

THEOREM 2 There are $C(n+r-1,r) = C(n+r-1,n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

EXAMPLE 4 Suppose that a cookie shop has four different kinds of cookies. How many ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem 2 this equals $C(9,6) = C(9,3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$

EXAMPLE 5 How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Solution: To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements so that x_1 items of type one, x_2 items of type two, and x_3 items of type three are chosen. Hence the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements. From Theorem 2 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{7 \cdot 6}{1 \cdot 2} = 21$$

EXAMPLE 6 What is the value of k after the following pseudocode has been executed?

```
k:=0
for i_1 := 1 to n
    for i_2 := 1 to i_1
        .
        .
        .
        for i_m := 1 to i_{m-1}
            k:=k+1
```

Solution: Note that the initial value of k is 0 and that 1 is added to k each time the nested loop is traversed with a sequence of integers $i_1, i_2, \cdots, i_m$ such that

$$1 \leq i_m \leq i_{m-1} \leq \cdots \leq i_1 \leq n$$

The number of such sequences of integers is the number of ways to choose m integers from $\{1,2,\cdots,n\}$, with repetition allowed. (To see this, note that once such a sequence has been selected, if we order the integers in the sequence in nondecreasing order, this uniquely defines an assignment of $i_m, i_{m-1}, \cdots, i_1$. Conversely, every such assignment corresponds to a unique unordered set.) hence, from Theorem 2, it follows that $k = C(n+m-1, m)$ after this code has been executed.

TABLE 1 Combinations and Permutations with and without repetition		
Type	**Repetition allowed?**	**Formula**
r-permutations	No	$P(n,r) = \dfrac{n!}{(n-r)!}$
r-combinations	No	$C(n,r) = \dfrac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\dfrac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

EXAMPLE 7 How many different strings can be made by reordering the letters of the word

SUCCESS?

Solution: Because some of the letters of *SUCCESS* are the same, the answer is *not* given by the number of permutations of seven letters. This word contains three *S*s, two *C*s, one *U*, and one *E*. To determine the number of different strings that can be made by reordering the letters, first note that the three *S*s can be placed among the seven positions in $C(7,3)$ ways, leaving four positions free. Then the two *C*s can be placed in $C(4,2)$ ways, leaving two free positions. The *U* can be placed in $C(2,1)$ ways, leaving just one position free. Hence *E* can be placed $C(1,1)$ way. Consequently, from the product rule, the number of different strings that can be made is

$$C(7,3) \cdot C(4,2) \cdot C(2,1) \cdot C(1,1) = \frac{7!}{3!\,4!} \cdot \frac{4!}{2!\,2!} \cdot \frac{2!}{1!\,1!} \cdot \frac{1!}{1!\,0!} = \frac{7!}{3!\,2!\,1!\,1!}$$
$$= 420$$

THEOREM 3 The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …, and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Distributing Objects into Boxes

- Many counting problems can be solved by enumerating the ways objects can be placed into boxes. The objects can be either *distinguishable*, that is, different from each other, or *indistinguishable*, that is, considered identical.
- Distinguishable objects are sometimes said to be *labeled*, whereas indistinguishable objects are said to be *unlabeled*.

DISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES. We first consider the case when distinguishable objects are placed into distinguishable boxes.

EXAMPLE 8 How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?
Solution: We will use the product rule to solve this problem.

To begin, note that the first player can be dealt 5 cards in $C(52,5)$ ways.

The second player can be dealt 5 cards in $C(47,5)$ ways, because only 47 cards are left.

The third player can be dealt 5 cards in $C(42,5)$ ways.

Finally, the fourth player can be dealt 5 cards in $C(37,5)$ ways.

Hence, the total number of ways to deal four players 5 cards each is:

$$C(52,5)\cdot C(47,5)\cdot C(42,5)\cdot C(37,5) = \frac{52!}{47!\,5!}\cdot\frac{47!}{42!\,5!}\cdot\frac{42!}{37!\,5!}\cdot\frac{37!}{32!\,5!}$$
$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}$$

Remark: The solution to Example 8 equals the number of permutations of 52 objects, with 5 indistinguishable objects of each of four different types, and 32 objects of a fifth type.