## Introduction

As we remarked in the previous lecture, the number $r$-combinations from a set with $n$ elements is often denoted by $\binom{n}{r}$. This number is also called a binomial coefficient because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a+b)^n$. We will discuss the binomial theorem, which gives a power of a binomial expression as a sum of terms involving binomial coefficients.

## The Binomial Theorem

**EXAMPLE 1** Produce the expansion of $(x+y)^3$ by employing combinatorial reasoning instead of finding the product of the three terms.

*Solution*: When $(x+y)^3 = (x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form $x^3, x^2y, xy^2$, and $y^3$ arise. To obtain a term of the form $x^3$, an $x$ must be chosen in each of the sums, and this can be done in only one way. Thus, the $x^3$ term in the product has a coefficient of 1.

To obtain a term of the form $x^2y$, an $x$ must be chosen in two of the three sums (and consequently a $y$ in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely $\binom{3}{2}$.

Similarly, the number of terms of the form $xy^2$ is the number of ways to pick one of the three sums to obtain an $x$ (and consequently take a $y$ from each of the other two sums). This can be done in $\binom{3}{1}$ ways.

Finally, the only way to obtain a $y^3$ term is to choose the $y$ for each of the three sums in the product, and this can be done in exactly one way. Consequently, it follows that

- $(x+y)^3 = (x+y)(x+y)(x+y)$
- $= (xx+xy+yx+yy)(x+y)$
- $= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy$
- $= x^3+3x^2y+3xy^2+y^3$

**THEOREM 1** **THE BINOMIAL THEOREM**. Let $x$ and $y$ be variables, and let $n$ be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n$$

**EXAMPLE 2** What is the expansion of $(x+y)^4$?

*Solution*:

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$
$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$
$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

**EXAMPLE 3** What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

*Solution*: From the Binomial Theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{\mathbf{25!}}{\mathbf{13!\,12!}} = 5{,}200{,}300$$

**EXAMPLE 4** What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

*Solution*: First, note that this expression equals $(2x+(-3y))^{25}$. By the Binomial Theorem, we have

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{\mathbf{25!}}{\mathbf{13!\,12!}} \mathbf{2^{12} 3^{13}}$$

**COROLLARY 1** Let $n$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

**COROLLARY 2** Let $n$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

**Remark**

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

**COROLLARY 3** Let $n$ be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

## Pascal's Identity and Triangle

**THEOREM 2** **PASCAL'S IDENTITY.** Let $n$ and $k$ be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$