Introduction

Good morning. Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the **order** of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the **order** of these elements **does not matter**. For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students? In this lecture we will develop methods to answer questions such as these.

Permutations

EXAMPLE 1.1 In how many ways can we select three students from a group of five students to stand in line for a picture?

Solution: First, note that the **order** in which we select the students **matters**. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $\mathbf{5 \cdot 4 \cdot 3 = 60}$ ways to select three students from a group of five students to stand in line for a picture.

EXAMPLE 1.2 In how many ways can we arrange all five of these students in a line for a picture?

Solution: To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways and the fifth in one way. Consequently, there are $\mathbf{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120}$ ways to arrange all five students in a line for a picture.

Definition A **permutation** of a set of distinct objects is an **ordered** arrangement of these objects.

An ordered arrangement of r elements of a set is called an $\boldsymbol{r}$**-permutation**.

The number of r-permutations of a set with n elements is denoted by $\boldsymbol{P(n,r)}$.

We can find $P(n,r)$ using the product rule.

EXAMPLE 2 Let $S = \{1,2,3\}$. The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3,2 is a 2-permutation of S.

THEOREM 1 If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

COROLLARY 1 If n and r are integers with $0 \le r \le n$, then

$$\boldsymbol{P(n,r) = \frac{n!}{(n-r)!}}$$

EXAMPLE 4 How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

EXAMPLE 7 How many permutations of the letters $ABCDEFGH$ contain the string ABC?

Solution: Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters $ABCDEFGH$ in which ABC occurs as a block.

Combinations

EXAMPLE 8 How many different committees of three students can be formed from a group of four students?

Solution: To answer this question, we need only find the number of subsets with three elements from the set containing four students. We see that there are four such subsets, one for each of the four students, because choosing four students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the **order** in which these students are chosen **does not matter**.

Definition An ***r*-combination** of elements of a set is an **unordered** selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

The number of r-combinations of a set with n distinct elements is denoted $\boldsymbol{C(n,r)}$. Note that $C(n,r)$ is also denoted by $\binom{\boldsymbol{n}}{\boldsymbol{r}}$ and is called a binomial coefficient.

EXAMPLE 10 How many subsets of the set $S = \{a, b, c, d\}$ are there that have only two elements?

Solution: Since order is not important we find $C(4,2) = 6$ consisting of the sets $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}$, and $\{c, d\}$.

THEOREM 2 The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$ is

$$\boldsymbol{C(n,r) = \frac{n!}{r!\,(n-r)!}}$$

EXAMPLE 11.1 How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Solution: Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are

$$C(52,5) = \frac{52!}{5!\,47!}$$

different hands of five cards that can be dealt. To compute the value $C(52,5)$, first divide numerator and denominator by 47! to obtain

$$C(52,5) = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960.$$

EXAMPLE 11.2 How many ways are there to select 47 cards from a standard deck of 52 cards

Solution: Because the order in which the 47 cards are dealt from a deck of 52 cards does not matter, there are

$$C(52,47) = \frac{52!}{47!\,5!}$$

different ways to select 47 cards from a standard deck of 52 cards.

3. How many different permutations of the set $\{a, b, c, d, e, f, g\}$ end with a?
Solution:
If we want the permutation to end with a, then we may as well forget about the a, and just count the number of permutations of $\{b, c, d, e, f, g\}$. Each permutation of these 6 letters, followed by a, will be a permutation of the desired type, and conversely. Therefore, the answer is:

$$P(6,6) = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{720}{1} = 720$$

9. How many possibilities are there for the win, place and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?
Solution:
We need to pick 3 horses from the 12 horses in the race, and we need to arrange them in order (first, second, and third), in order to specify the win, place, and show. Thus there are:

$$P(12,3) = \frac{12!}{(12-3)!} = \frac{12!}{9!} = 12 \cdot 11 \cdot 10 = 1320$$

possibilities.

11. How many bit strings of length 10 contain

a) exactly four 1s?
Solution:
To specify a bit string of length 10 that contains exactly four 1's, we simply need to choose the four positions that contain the 1's. There are $C(10,4) = 210$ ways to do that.

$$C(10,4) = 210$$

b) at most four 1s?
Solution:
To contain at most four 1's means to contain four 1's three 1's two 1's one 1 or no 1's. Reasoning as in part (a), we see that there are:

$$C(10,4) + C(10,3) + C(10,2) + C(10,1) + C(10,0)$$
$$= 210 + 120 + 45 + 10 + 1$$
$$= 386$$

to do that.

c) at least four 1s?
Solution:
To contain at least four 1's means to contain four 1's, , five 1's, six 1's, seven 1's, eight 1's, nine 1's, or 10 1's. We could reason as in part (b), but we would have many numbers to add. A simpler approach would be to figure out the number of ways not to have at least four 1's (i.e., to have three 1's, two 1's, one 1, or no 1's) and then subtract that from 2^{10} , the total number of bit strings of length 10. This way we get $1024 - (120 + 45 + 10 + 1) = 848$

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d) an equal number of 0s and 1s?
Solution:

To have an equal number of 0's and 1's in this case means to have five 1's. Therefore the answer is $C(10,5) = 252$.

$$C(10,5) = 252$$

18. A coin is flipped 8 times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?

Solution:

Each flip can be either heads or tails, so there are $2^8 = 256$ possible outcomes.

$$2^8 = 256$$

b) contain exactly three heads?

Solution:

To specify an outcome that has exactly three heads, we simply need to choose the three flips that came up heads. There are $C(8,3) = 56$ such outcomes.

$$C(8,3) = 56$$

c) contain at least three heads?

Solution:

To contain at least three heads means to contain three heads, four heads, five heads, six heads, seven heads, or eight heads. Reasoning as in part (b), we see that there are $C(8,3) + C(8,4) + C(8,5) + C(8,6) + C(8,7) + C(8,8) = 56 + 70 + 56 + 28 + 8 + 1 = 219$ such outcomes. We could also subtract from 256 the number of ways to get two or fewer heads, namely $28 + 8 + 1 = 37$. Since $256 - 37 = 219$, we obtain the same answer using this alternative method.

$$C(8,3) + C(8,4) + C(8,5) + C(8,6) + C(8,7) + C(8,8) = 219$$

d) contain the same number of heads and tails?

Solution:

To have an equal number of heads and tails in this case means to have four heads. Therefore the answer is $C(8,4) = 70$.

$$C(8,4) = 70$$