Introduction

Good morning. Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, at least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it. Of course, this principle applies to other objects besides pigeons and pigeonholes.

THEOREM 1 **THE PIGEONHOLE PRINCIPLE**. If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

COROLLARY 1 A function f from a set with $k + 1$ or more elements to set with k elements is not one-to-one.

EXAMPLE 1 Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

EXAMPLE 2 In any group of 27 English words, there must be at least two with the same first letter because there are 26 letters in the English alphabet.

EXAMPLE 3 How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?
Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Generalized Pigeonhole Principle

THEOREM 2 **THE GENERALIZED PIGEONHOLE PRINCIPLE**. If N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects.

EXAMPLE 5 Among 100 people there are least $\lceil 100/12 \rceil = 9$ who were born in the same month.

EXAMPLE 6 What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade.

EXAMPLE 7.1 How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
Solution: Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $2 \cdot 4 + 1 = 9$, so nine cards suffice. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed. Consequently, nine cards must be selected to guarantee that at least three cards of one suit are chosen. One good way to think about this is to note that after the eighth card is chosen, there is no way to avoid having a third card of some suit.

EXAMPLE 7.2 How many cards must be selected to guarantee that at least three hearts are selected?
Solution: We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.

EXAMPLE 8 What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form NXX-NXX-XXXX, where the first three digits form the area code. N represents a digit from 2 to 9 and X represents any digit.
Solution: There are eight million different phone numbers of the form NXX-XXXX (as shown in Example 8 of Section 6.1). Hence, by the generalized pigeonhole principle, among 25 million telephones, at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different.

Some Elegant Applications of the Pigeonhole Principle

EXAMPLE 10 During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \cdots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover $a_1 + 14, a_2 + 14, \cdots, a_{30} + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$.

The 60 positive integers $a_1, a_2, \cdots, a_{30}, a_1 + 14, a_2 + 14, \cdots, a_{30} + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers $a_j, j = 1, 2, \cdots, 30$ are all distinct and the integers $a_j + 14, j = 1, 2, \cdots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i.

EXAMPLE 11 Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Solution: Write each of the $n + 1$ integers $a_1, a_2, \cdots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for , $j = 1, 2, \cdots, n + 1$, where k_j is a nonnegative integer and q_j is odd.

THEOREM 3 Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

EXAMPLE 12 The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains 10 terms. Note that $10 = 3^2 + 1$. There are four increasing subsequences of the length four, namely 1, 4, 6, 12; 1, 4, 6, 7; 1, 4, 6, 10; and 1, 4, 5, 7. There is also a decreasing subsequence of length four, namely 11, 9, 6, 5.

EXAMPLE 13 Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Solution: Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from the generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements. In the former case, suppose that B, C, and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise, B, C, and D form a set of three mutual enemies. The proof in the latter case, when there are three or more enemies of A, proceeds in a similar manner.

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on the weekends.

Solution:

There are six classes: these are the pigeons. There are five days on which classes may meet (Monday through Friday): these are the pigeonholes. Each class must meet on a day (each pigeon must occupy a pigeonhole). By the pigeonhole principle at least one day must contain at least two classes.

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

Solution:

There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains two pigeons. By the pigeonhole principle, the answer is 3. If three socks are taken from the drawer, at least two must have the same color. On the other hand, two socks are not enough, because one might be brown and the other black. Note that the number of socks was irrelevant (assuming it was at least 3).

b) How many socks must he take out to be sure that he has at least two black socks?

Solution:

He needs to take out 14 socks in order to insure at least two are black socks. If he does so, then at most 12 of them are brown, so at least two are black. On the other hand, if he removes 13 or fewer socks, then 12 of them could be brown, and he might not get his pair of black socks. This time the number of socks did matter.

7. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

Solution:

Let the n consecutive integers be denoted $x+1, x+2, \ldots, x+n$, where x is some integer. We want to show that exactly one of these is divisible by n. There are n possible remainders when an integer is divided by n, namely $0, 1, 2, \ldots, n-1$. There are two possibilities for the remainders in our collection of n numbers: either they cover all the possible remainders (in which case exactly one of our numbers has a remainder of 0 and is therefore divisible by n), or they do not. If they do not, then by the pigeonhole principle, since there are then fewer than n pigeonholes (remainders) for n pigeons (the numbers in our collection), at least one remainder must occur twice. In other words, it must be the case that $x+i$ and $x+j$ have the same remainder when divided by n for some pair of numbers i and j with $0 < i < j \leq n$. Since $x+i$ and $x+j$ have the same remainder when divided by n, if we subtract $x+i$ from $x+j$, then we will get a number divisible by n. This means that $j-i$ is divisible by n. But this is impossible, since $j-i$ is a positive integer strictly less than n. Therefore the first possibility must hold, that exactly one of the numbers in our collection is divisible by n.

Like 12. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \ \mathbf{mod}\ 3 = a_2 \ \mathbf{mod}\ 3$ and $b_1 \mathbf{mod}\ 3 = b_2 \ \mathbf{mod}\ 3$?

Solution:

Working modulo 3 there are 9 pairs: (0, 0), (0, 1) . . ., (2, 2). Thus we could have 9 ordered pairs of integers (a, b) such that no two of them were equal when reduced modulo 3. The pigeonhole principle, however, guarantees that if we have 10 such pairs, then at least two of them will have the same coordinates, modulo 3.