Introduction

Good morning. Combinatorics is the study of arrangements of objects. Perhaps, the first application of the study of combinatorics was in the study of gambling games. Understanding combinatorics helps us determine the time complexity of certain algorithms. Counting defines the requirements for the number of telephone numbers or the number of internet protocol (IP) addresses. Counting techniques are used extensively when probabilities of events are computed.

Basic Counting Principles

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task **and for each of these ways** of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

EXAMPLE 1 A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

EXAMPLE 2 The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution: The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600.

EXAMPLE 3 There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

Solution: The procedure of choosing a port consists of two tasks, first picking a microcomputer and then picking a port on this microcomputer. Because there are 32 ways to choose the microcomputer and 24 ways to choose the port no matter which microcomputer has been selected, the product rule shows that there are $32 \cdot 24 = 768$ ports.

EXAMPLE 4 How many different bit strings of length seven are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either a 0 or a 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven.

EXAMPLE 5 How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequence of letters are prohibited, even if they are obscene)?

Solution: There are 26 choices for each of the three uppercase English letters and ten choices for each of the three digits. Hence by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates.

EXAMPLE 6 **Counting Functions.** How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements.

EXAMPLE 7 **Counting One-to-One Functions.** How many one-to-one functions are there from a set with m elements to a set with n elements?

Solution: First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Now let $m \le n$. Suppose the elements in the domain are $a_1, a_2, \cdots, a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at a_k can be chosen in $n - k + 1$ ways. By the product rule, there are $n(n-1)(n-2)\cdots(n-m+1)$ one-to-one functions from a set with m elements to one with n elements.

For example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements.

THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where **none** of the set of n_1 ways is the same as **any** of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

EXAMPLE 12 Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and **no one is both a faculty member and a student**?

Solution: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

EXAMPLE 13 A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. **No project is on more than one list**. How many possible projects are there to choose from?
Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. **Because no project is on more than one list,** by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

EXAMPLE 14 What is the value of k after the following code, where $n_1, n_2, \cdots, n_m$ are positive integers, has been executed?

$k := 0$
for $i_1 :=$ **1 to** n_1
    $k := k + 1$
for $i_2 :=$ **1 to** n_2
    $k := k + 1$
        ⋮
for $i_m :=$ **1 to** n_m
    $k := k + 1$

Solution: The initial value of k is zero. This block of code is made up of m different loops. Each time a loop is traversed, 1 is added to k. To determine the value of k after this code has been executed, we need to determine how many times we traverse a loop. Note that there are n_i ways to traverse the ith loop. Because **we only traverse one loop at a time**, the sum rule shows that the final value of k, which the number of ways to traverse one of the m loops is $n_1 + n_2 + \cdots + n_m$.

More Complex Counting Problems

EXAMPLE 15 In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits). Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there?
Solution: Let V equal the number of different variable names in this version of BASIC. Let V_1 be the number of these that are one character long and V_2 be the number of these that are two characters long. Then the sum rule $V = V_1 + V_2$. Note that $V_1 = 26$, because a one-character variable name must be a letter. Furthermore, by the product rule there are $26 \cdot 36$ strings of length two that begin with a letter and end with an alphanumeric character. However, five of these are excluded, so $V_2 = 26 \cdot 36 - 5 = 931$. Hence there are $V = V_1 + V_2 = 26 + 931 = 957$ different names for variables in this version of BASIC.

The Subtraction Rule (Inclusion – Exclusion for Two Sets)

> **THE SUBTRACTION RULE** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

How many elements are there in the union of sets A_1 and A_2?

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

EXAMPLE 18 How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00.

We can construct a bit string of length eight that begins with a 1 in $2^7 = 128$ ways. This follows by the product rule, because the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways.

We can construct a bit string of length eight that ends with the two bits 00 in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.

Some of the ways to construct a bit string of length eight starting with 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5 = 32$ ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in only one way. Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00 equals $128 + 64 - 32 = 160$.

The Division Rule

THE DIVISION RULE There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

EXAMPLE 20 How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Solution: We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that there are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and 1 way to select the person for seat 4. There are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table.

Tree Diagrams

Counting problems can be solved using tree diagrams. A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches. To use trees in counting, we use a branch to represent each possible choice. We present the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.

EXAMPLE 21 How many bit strings of length four do not have two consecutive 1s?

Solution: The tree diagram in Figure 2 displays all bit strings of length four without two consecutive 1s. We see that there are eight bit strings of length four without two consecutive 1s.

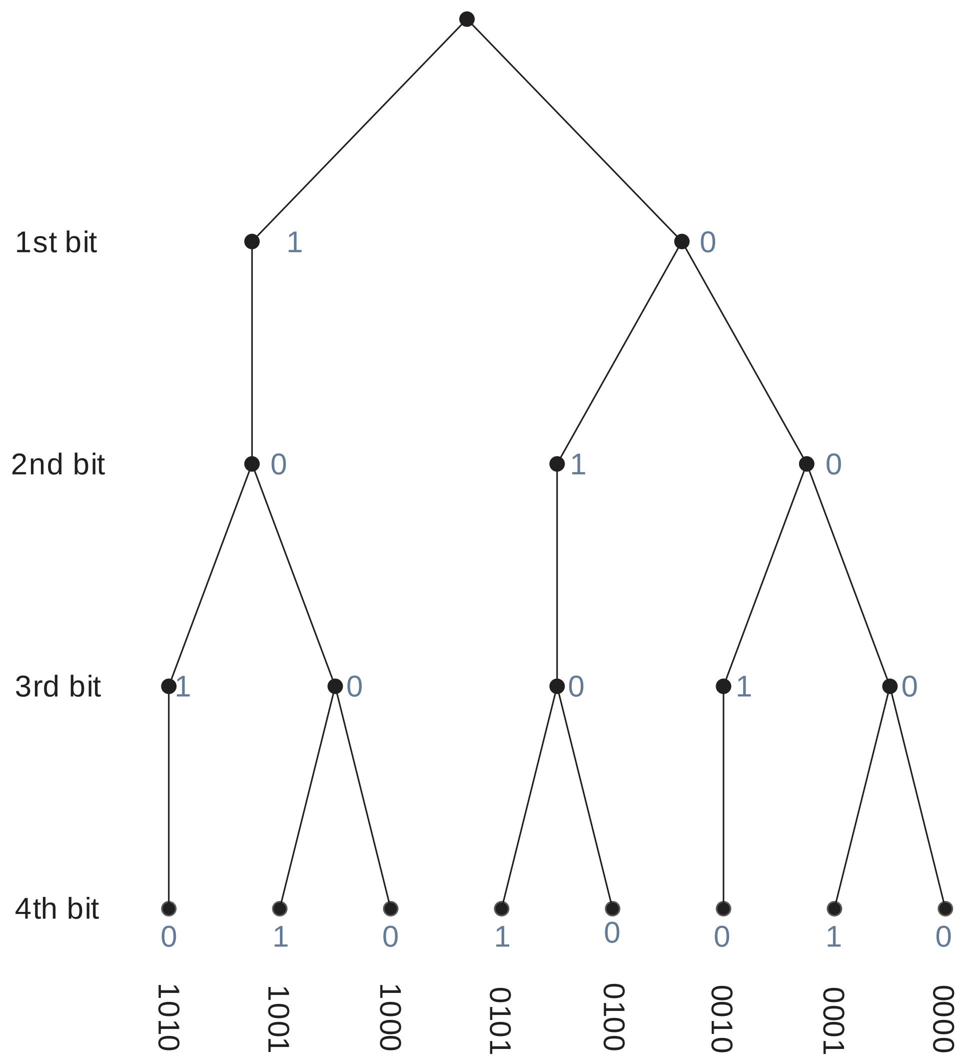

FIGURE 2 Bit Strings of Length Four without Consecutive 1s

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?

   Solution:
   Apply the product rule: $6 \times 7 = 42$.