Introduction

recursion A definition that defines an object in terms of itself.

Recursively Defined Functions

Two steps are employed to define a function with the set of nonnegative integers as its domain:

1. *BASIS STEP*: Specify the value of the function at zero.
2. *RECURSIVE STEP*: Give a rule for finding its value at an integer from its values at smaller integers.

EXAMPLE 1 Suppose that f is defined recursively by

$f(0) = 3.$

$f(n+1) = 2f(n) + 3.$

Find $f(1)$, $f(2)$, $f(3)$ and $f(4)$.

Solution:

$f(1) =$	$2f(0) + 3$	9
$f(2) =$	$2f(1) + 3$	21
$f(3) =$	$2f(2) + 3$	45
$f(4) =$	$2f(3) + 3$	93

EXAMPLE 2 Give an inductive definition of the factorial function $F(n) = n!$.

Solution:

1. *BASIS STEP*: $F(0) = 1$
2. *RECURSIVE STEP:* $F(n+1) = (n+1) \cdot F(n)$

EXAMPLE 3 Give an inductive definition of the function a^n, where a is a nonzero real number an n is a nonnegative integer.

Solution:

1. *BASIS STEP*: $f(0) = 1$, $a^0 = 1$
2. *RECURSIVE STEP:* $f(n+1) = a \cdot f(n)$, $a^{n+1} = a \cdot a^n$

EXAMPLE 4 Give a recursive definition of

$$\sum_{k=0}^{n} a_k$$

Solution:

1. $\sum_{k=0}^{0} a_k = a_0$
2. $\sum_{k=0}^{n+1} a_k = \left(\sum_{k=0}^{n} a_k\right) + a_{n+1}$

DEFINITION 1 The *Fibonacci numbers*, $f_0, f_1, f_2, \cdots$, are defined by the equations $f_0 = 0$ and $f_1 = 1$, and

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \cdots$

EXAMPLE 5 Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution:

$f_2 = f_1 + f_0 =$	$1 + 0 =$	1
$f_3 = f_2 + f_1 =$	$1 + 1 =$	2
$f_4 = f_3 + f_2 =$	$2 + 1 =$	3
$f_5 = f_4 + f_3 =$	$3 + 2 =$	5
$f_6 = f_5 + f_4 =$	$5 + 3 =$	8

5. Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and **prove** that your formula is valid.

a) $f(0) = 0, f(n) = 2f(n-2)$ for $n \geq 1$

Solution:

n	0	1
$f(n)$	0	$2f(1-2) = 2f(-1)$

Function f is defined only from the set of nonnegative integers implying that the argument for f must be a nonnegative integer. When $n = 1$, $f(n) = 2f(-1)$. $f(-1)$ is not defined since -1 is not a nonnegative integer.

b) $f(0) = 1, f(n) = f(n-1) - 1$ for $n \geq 1$

Solution:

n	0	1	2	3
$f(n)$	1	$f(n-1)-1$ $= f(1-1)-1$ $= f(0)-1$ $= 1-1$ $= 0$	$f(n-1)-1$ $= f(2-1)-1$ $= f(1)-1$ $= 0-1$ $= -1$	$f(3-1)-1$ $= f(3-1)-1$ $= f(2)-1$ $= -1-1$ $= -2$

Conjecture: $f(n) = 1 - n$

Basis Step: $f(0) = 1 = 1 - n = 1 - 0 = 1$

Inductive Step:

Induction Hypothesis: $f(k) = 1 - k$

Prove: $f(k+1) = 1 - (k+1)$

Expression	Justification
$f(k+1)$	Original left hand side
$= f(k) - 1$	Application of original definition $f(n) = f(n-1) - 1$
$= 1 - k - 1$	Application of the induction hypothesis $f(k) = 1 - k$
$= 1 - (k+1)$	$-k - 1 = -(k+1)$

c) $f(0) = 2, f(1) = 3, f(n) = f(n-1) - 1$ for $n \geq 2$

Solution:

n	0	1	2	3
$f(n)$	2	$f(1) = 3$	$f(2)$ $= f(2-1) - 1$ $= f(1) - 1$ $= 3 - 1$ $= 2$	$f(3)$ $= f(3-1) - 1$ $= f(2) - 1$ $= 2 - 1$ $= 1$

Conjecture: $f(0) = 2$ for $n = 0$; $f(n) = 4 - n$ for $n > 0$

Basis Step:

For $n = 0$: $f(0) = 2$

For $n > 0$: $f(1) = 4 - 1 = 3 = f(1)$

Inductive Step:

Induction Hypothesis: $f(k) = 4 - k$

Prove: $f(k+1) = 4 - (k+1)$

Expression	Justification
$f(k+1)$	Original left hand side
$= f(k) - 1$	Application of original definition $f(n) = f(n-1) - 1$
$= 4 - k - 1$	Application of the induction hypothesis $f(k) = 4 - k$
$= 4 - (k+1)$	$-k - 1 = -(k+1)$

d) $f(0) = 1, f(1) = 2, f(n) = 2f(n-2)$ for $n \geq 2$

Solution:

n	0	1	2	3	4	5	6
$f(n)$	1	2	$2f(2-2)$ $= 2f(0)$ $= 2 \times 1$ $= 2$	$2f(3-2)$ $= 2f(1)$ $= 2 \times 2$ $= 4$	$2f(4-2)$ $= 2f(2)$ $= 2 \times 2$ $= 4$	$2f(5-2)$ $= 2f(3)$ $= 2 \times 4$ $= 8$	$2f(6-2)$ $= 2f(4)$ $= 2 \times 4$ $= 8$

Conjecture: $f(n) = 2^{\lfloor (n+1)/2 \rfloor}$

Basis Step:

$f(0) = 1 = 2^{\lfloor (n+1)/2 \rfloor} = 2^{\lfloor (0+1)/2 \rfloor} = 2^{\lfloor (1)/2 \rfloor} = 2^{0} = 1$

$f(1) = 2 = 2^{\lfloor (n+1)/2 \rfloor} = 2^{\lfloor (1+1)/2 \rfloor} = 2^{\lfloor (2)/2 \rfloor} = 2^{1} = 2$

Inductive Step:

Induction Hypothesis: $f(k) = 2^{\lfloor (k+1)/2 \rfloor}$

Prove: $f(k+1) = 2^{\lfloor ((k+1)+1)/2 \rfloor}$

Expression	Justification
$f(k+1)$	Original left hand side
$= 2f(k+1-2)$	Application of original definition $f(n) = 2f(n-2)$
$= 2f(k-1)$	Subtraction
$= 2 \times 2^{\lfloor (k-1+1)/2 \rfloor}$	Application of the induction hypothesis $f(k) = 2^{\lfloor (k+1)/2 \rfloor}$
$= 2 \times 2^{\lfloor k/2 \rfloor}$	Subtraction
$= 2^{\lfloor k/2 \rfloor + 1}$	Properties of exponents: $2 \times 2^{x} = 2^{x+1}$
$= 2^{\lfloor k/2 \rfloor + \lfloor 2/2 \rfloor}$	$1 = \frac{2}{2}$
$= 2^{\lfloor (k+2)/2 \rfloor}$	$\frac{k}{2} + \frac{2}{2} = \frac{k+2}{2}$
$= 2^{\lfloor ((k+1)+1)/2 \rfloor}$	$\frac{k+2}{2} = \frac{((k+1)+1)}{2}$

e) $f(0) = 1, f(n) = 3f(n-1)$ if n is odd and $n \geq 1$
and $f(n) = 9f(n-2)$ if n is even and $n \geq 2$

Solution:

n	0	1	2	3	4	5
$f(n)$	1	$3f(1-1)$ $= 3f(0)$ $= 3 \times 1$ $= 3$		$3f(3-1)$ $= 3f(2)$ $= 3 \times 9$ $= 27$		$3f(5-1)$ $= 3f(4)$ $= 3 \times 81$ $= 243$
$f(n)$			$9f(2-2)$ $= 9f(0)$ $= 9 \times 1$ $= 9$		$9f(4-2)$ $= 9f(2)$ $= 9 \times 9$ $= 81$	

Conjecture: $f(n) = 3^n$

Basis Step:

case:

$n = 0$: $f(0) = 1 = 3^0 = 1$

$n = 1$: $f(1) = 3f(1-1) = 3f(0) = 3 \times 1 = 3 = 3^1 = 3$

$n = 2$: $f(2) = 9f(2-2) = 9f(0) = 9 \times 1 = 9 = 3^2 = 9$

Inductive Step:

Assume $f(k) = 3^k$

Prove: $f(k+1) = 3^{k+1}$

case:

For odd n:

Expression	Justification
$f(n) = 3f(n-1)$	From the exercise statement when n is odd and $n \geq 1$
$= 3 \times 3^{n-1}$	Apply the induction hypothesis
$= 3^n$	Properties of exponents: $3 \times 3^x = 3^{x+1}$

For even $n > 1$:

Expression	Justification
$f(n) = 9f(n-2)$	From the exercise statement when n is odd and $n \geq 1$
$= 9 \times 3^{n-2}$	Apply the induction hypothesis
$= 3^2 \times 3^{n-2} = 3^n$	Properties of exponents: $3 \times 3^x = 3^{x+1}$