Mathematical Induction

- **Mathematical induction** can be used to prove statements that assert $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical induction has two parts
  - a **basis step**, where we show $P(1)$ is true and
  - an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION. To prove $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps.

1. *BASIS STEP*. We verify that $P(1)$ is true.
2. *INDUCTIVE STEP*. We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Strong Induction

STRONG INDUCTION. To prove $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps.

1. *BASIS STEP*. We verify that $P(1)$ is true.
2. *INDUCTIVE STEP*. We show that the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \to P(k+1)$ is true for all positive integers k.

EXAMPLE 2

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Solution: Let $P(n)$ be the proposition that n can be written as the product of primes.

BASIS STEP: $P(2)$ is true, because 2 can be written as the product of one prime, itself. [Note that $P(2)$ is the first case we need to establish.]

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(j)$ is true for all positive integers $j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k. To complete the inductive step, it must be shown that $P(k+1)$ is true under this assumption, that is, that $k+1$ is the product of primes.

There are two cases to consider, namely, when $k+1$ is prime and when $k+1$ is composite. If $k+1$ is prime, we immediately see that $P(k+1)$ is true.

Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b \leq k+1$. By the inductive hypothesis, both a and b can be written as the product of primes. Thus, if $k+1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.

EXAMPLE 4 Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution: We will prove this result using the principle of mathematical induction. Then we will present a proof using strong induction.

Let $P(n)$ be the statement that postage of n cents can be formed using 4-cent and 5-cent stamps.

We begin by using the principle of mathematical induction.

BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true. That is, under this hypothesis, postage of k cents can be formed using 4-cent and 5-cent stamps. To complete the inductive step, we need to show that when we assume $P(k)$ is true, then $P(k+1)$ is also true where $k \geq 12$. That is, we need to show that if we can form postage of k cents, then we can form postage of $k+1$ cents.

To see this, suppose that at least one 4-cent stamp was used to form postage of k cents. Then we can replace this stamp with a 5-cent stamp to form postage of $k+1$ cents.

But if no 4-cent stamps were used, we can form postage of k cents using only 5-cent stamps. So we can replace three 5-cent stamps with four 4-cent stamps to form postage of $k+1$ cents. This completes the inductive step.

STRONG INDUCTION:

BASIS STEP: We show that $P(12), P(13), P(14)$, and $P(15)$ are true. We can form postage of 12, 13, 14, and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively. This shows that $P(12), P(13), P(14)$, and $P(15)$ are true.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for $12 \leq j \leq k$, where k is an integer with $k \geq 15$. That is, we assume that we can form postage of *j* cents, where $12 \leq j \leq k$.

To complete the inductive step we need to show that under this assumption, $P(k+1)$ is true, that is, we can form postage of $k+1$ cents. Using the inductive hypothesis, we can assume that $P(k-3)$ is true because $k-3 \geq 12$, that is, we can form postage of $k-3$ cents using just 4-cent and 5-cent stamps. To form postage of $k+1$ cents, we need only add another 4-cent stamp to the stamps we used to form postage of $k-3$ cents. That is, we have shown that if the inductive hypothesis is true, then $P(k+1)$ is also true. This completes the inductive step.