Mathematical Induction

- **Mathematical induction** can be used to prove statements that assert $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function. A proof by mathematical induction has two parts
  - a **basis step**, where we show $P(1)$ is true and
  - an **inductive step**, where we show that for all positive integers k, if $P(k)$ is true, then $P(k+1)$ is true.

Mathematical Induction

PRINCIPLE OF MATHEMATICAL INDUCTION. To prove $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps.

1. *BASIS STEP*. We verify that $P(1)$ is true.
2. *INDUCTIVE STEP*. We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Remark
Expressed as a rule of inference, this proof technique can be stated as

$$[P(1) \wedge \forall k(P(k) \rightarrow P(k+1))] \rightarrow \forall n P(n)$$

Remark
In a proof by mathematical induction it is *not* assumed that $P(k)$ is true for all positive integers! It is only shown that *if it is assumed* that $P(k)$ is true, then $P(k+1)$ is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

EXAMPLE 1
Show that if n is a positive integer, then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Solution:

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Now, we must prove that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

1. Apply the inductive hypothesis to the left side

$$\frac{k(k+1)}{2} + (k+1)$$

2. Algebraically transform our result until it matches the right hand side

$$\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$\frac{k(k+1) + 2(k+1)}{2}$$

$$\frac{(k+2)(k+1)}{2}$$

$$\frac{(k+1)(k+2)}{2}$$

EXAMPLE 2 Find a formula for the sum of the positive odd integers. Then prove that your formula is correct using mathematical induction.

Solution: After fussin' some, we find $\sum_{i=1}^{n}(2i-1)=n^2$. Please refer to fussing below.

Integer	**Sum**	**Result**
1	1	1
2	1+3	4
3	1+3+5	9
4	1+3+5+7	16
n	$\sum_{i=1}^{n}(2i-1)$	n^2

BASIS STEP: $P(1)$ is true, because $1=2(1)-1=1$

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1+3+\cdots+2k-1=k^2$$

Now, we must prove that

$$1+3+\cdots+2k-1+2(k+1)-1=(k+1)^2$$

1. Apply the inductive hypothesis to the left side
$$k^2+2(k+1)-1$$
2. Algebraically transform our result until it matches the right hand side
$$\begin{aligned}k^2+2(k+1)-1&=k^2+2k+2-1\\&=k^2+2k+1\\&=(k+1)^2\end{aligned}$$

EXAMPLE 3 Use mathematical induction to show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

Solution: Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 2^1 - 1 = 1$

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

Now, we must prove that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

1. Apply the inductive hypothesis to the left side

$$2^{k+1} - 1 + 2^{k+1} =$$

2. Algebraically transform our result until it matches the right hand side

$$2 \cdot 2^{k+1} - 1 =$$

$$2^{k+1+1} - 1 =$$

$$2^{k+2} - 1 = 2^{k+2} - 1$$

$$w^5$$