Introduction
In this lecture we introduce and explain the three functions that are typically used to characterize the growth of algorithms. \( O(f(n)) \), or big Oh, is a ceiling saying that the algorithm grows no faster than \( f(n) \). \( \Omega(g(n)) \) is a floor saying that the algorithm grows no slower than \( g(n) \). \( \Theta(f(n)) \) is the defining function for an algorithm. When we can find a ceiling and a floor that employ the same function \( f(n) \), we say that the algorithm is \( \Theta(f(n)) \).

Big-\( O \) Notation
DEFINITION 1. Let \( T \) and \( f \) be functions from the set of integers or the set of real numbers to the set of real numbers. We say that \( T(n) \) is \( O(f(n)) \) if there are positive constants \( n_0 \) and \( C \) such that
\[
|T(n)| \leq C|f(n)|
\]
whenever \( n > n_0 \) [This is read as “\( T(n) \) is big-oh of \( f(n) \).”]

- The constants \( C \) and \( n_0 \) in the definition of big-\( O \) notation are called witnesses to the relationship \( T(n) \) is \( O(f(n)) \).
- There are infinitely many witnesses to the relationship \( T(n) \) is \( O(f(n)) \).

Finding \( C \), \( n_0 \), and \( f(n) \) for \( T(n) \)
Steps: Assume \( T(n) = \frac{3}{2}n^2 + \frac{5}{2}n + 10 \)
1. Find \( f(n) \). Let \( f(n) \) be the fastest growing term in \( T(n) \) with its coefficient removed. \( f(n) = n^2 \)
2. Find \( C \).
   2.1. \( C = C_{\min} + \Delta \), where \( \Delta = 1 \) (in many cases).
   2.2. \( C_{\min} = \lim_{n \to \infty} \frac{T(n)}{f(n)} = \lim_{n \to \infty} \frac{\frac{3}{2}n^2 + \frac{5}{2}n + 10}{n^2} = \frac{3}{2} \)
   2.3. In practice, \( C_{\min} \) is the coefficient of the fastest growing term in \( T(n) \).
3. \( C = \Delta + \frac{3}{2} = 1 + \frac{3}{2} = \frac{5}{2} \)
4. Find \( n_0 \).
   4.1. Solve \( |T(n_0)| \leq C|f(n_0)| \)
\[
\frac{3}{2}n_0^2 + \frac{5}{2}n_0 + 10 \leq \frac{5}{2}n_0^2
\]
\[
n_0^2 - \frac{5}{2}n_0 - 10 \geq 0
\]
\[
n_0 = \frac{\frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 + 4 \cdot 1 \cdot 10}}{2}, n_0 > 0
\]
\[
n_0 = [4.65], n_0 > 0
\]
\[
n_0 = 5
\]
4.2. Choose an integer value for \( n_0 \). Let \( n_0 = 5 \).
We have shown that \( T(n) = \frac{3}{2}n^2 + \frac{5}{2}n + 10 \) is \( O(n^2) \) because we have found witnesses \( C = \frac{5}{2} \) and \( n_0 = 5 \).
\( T(n) \) is \( O(f(n)) \)

**Figure 1.** \( T(n) \) is \( O(f(n)) \)
As an estimate of the growth of $T(n)$, $O(f(n))$ is a conservative estimate because it places a ceiling on the growth $T(n)$. When $n > n_0$, $O(f(n)) \geq T(n)$. Let us call $O(f(n))$, big-O of $(n)$, the Eeyore estimate because the actual running time of a function, given by $T(n)$ will be less than the estimate $Cf(n)$ because Eeyore is a pessimist always promising less than he actually delivers.

Seek to emulate Eeyore.
DEFINITION 2

Let $T$ and $f$ be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $T(n)$ is $\Omega(g(n))$ if there are positive constants $n_0$ and $C$ such that

$$|T(n)| \geq C|g(n)|$$

whenever $n > n_0$.

[This is read as “$T(n)$ is big-Omega of $g(n)$.”]

Finding $C$, $n_0$, and $f(n)$ for $T(n)$

Steps: Assume $T(n) = \frac{3}{2}n^2 + \frac{5}{2}n - 10$

1. Find $g(n)$. Let $f(n)$ be the fastest growing term in $T(n)$ without its coefficient. $f(n) = n^2$

2. Find $C$.
   
   2.1. $C = C_{\min} - \Delta$, where $\Delta = 1$ (in many cases).

   2.2. $C_{\min} = \lim_{n \to \infty} \frac{T(n)}{g(n)} = \lim_{n \to \infty} \frac{\frac{3}{2}n^2 + \frac{5}{2}n - 10}{n^2} = \frac{3}{2}$

   2.3. In practice, $C_{\min}$ is the coefficient of the fastest growing term in $T(n)$.

   2.4. $C = \frac{3}{2} - \Delta = \frac{3}{2} - 1 = \frac{1}{2}$

3. Find $n_0$.
   
   3.1. Solve $|T(n_0)| \geq C|g(n_0)|$

   $$\frac{3}{2}n_0^2 + \frac{5}{2}n_0 - 10 \leq \frac{1}{2}n_0^2$$

   $$n_0^2 + \frac{5}{2}n_0 - 10 \leq 0$$

   $$n_0 = \left[\frac{-5 \pm \sqrt{(5)^2 + 4 \cdot 1 \cdot 10}}{2}\right], n_0 > 0$$

   $$n_0 = [2.15], n_0 > 0$$

   $$n_0 \geq 2$$

   3.2. For $n > 0$, $T(n) \geq Cg(n)$. Select $n_0 = 0$.

We have shown that $T(n) = \frac{3}{2}n^2 + \frac{5}{2}n - 10$ is $\Omega(n^2)$ because we have found witnesses $C = \frac{1}{2}$ and $n_0 = 2$. 
As an estimate of the growth of $T(n)\), \Omega(g(n)) is an optimistic estimate because it places a floor on the growth of $T(n)$. When $n > n_0$, $T(n) \geq \Omega(g(n))$. Let us call $\Omega(g(n))$, big-omega of $g(n)$, the Tigger estimate because the actual running time of a function, given by $T(n)$ will be greater than the estimate $Cg(n)$ and Tigger is an optimist always promising more than he can deliver.
Big-\(\Theta\) Notation

**DEFINITION 3**

Let \(T\) and \(f\) be functions from the set of integers or the set of real numbers to the set of real numbers. We say that \(T(n) = \Theta(f(n))\) if \(T(n) = O(f(n))\) and if \(T(n) = \Omega(f(n))\). When \(T(n) = \Theta(f(n))\), we say that \(T\) is big-Theta of \(f\) and we also say that \(T(n)\) is of order \(f\).

Let us find a function, \(T(n)\), that is both \(O(f(n))\) and \(\Omega(f(n))\).

**Figure 3.** \(T(n) = \Theta(f(n))\)