Introduction

In this lecture we introduce and explain the three functions that are typically used to characterize the growth of algorithms. $O\big(f(n)\big)$, or big Oh, is a ceiling saying that the algorithm grows no faster than $f(n)$. $\Omega\big(g(n)\big)$ is a floor saying that the algorithm grows no slower than $g(n)$. $\Theta\big(f(n)\big)$ is the defining function for an algorithm. When we can find a ceiling and a floor that employ the same function $f(n)$, we say that the algorithm is $\Theta\big(f(n)\big)$.

Big-O Notation

DEFINITION 1 *Let T and f be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $T(n)$ is $O(f(n))$ if there are positive constants n_0 and C such that*

$$|T(n)| \le C|f(n)|$$

whenever

$$n > n_0$$

[This is read as "$T(n)$ is big-oh of $f(n)$."]

- The constants C and n_0 in the definition of big-O notation are called **witnesses** to the relationship $T(n)$ is $O\big(f(n)\big)$.
- There are *infinitely many* witnesses to the relationship $T(n)$ is $O\big(f(n)\big)$.

Finding C, n_0, and $f(n)$ for $T(n)$

Steps: Assume $\boldsymbol{T(n) = \frac{3}{2}n^2 + \frac{5}{2}n + 10}$

1. Find $f(n)$. Let $f(n)$ be the fastest growing term in $T(n)$ with its coefficient removed.
   $\boldsymbol{f(n) = n^2}$
2. Find C.
   2.1. $C = C_{min} + \Delta$, where $\Delta = 1$ (in many cases).
   2.2. $C_{min} = \lim_{n\to\infty} \frac{T(n)}{f(n)} = \lim_{n\to\infty} \frac{\frac{3}{2}n^2+\frac{5}{2}n+10}{n^2} = \frac{3}{2}$
   2.3. In practice, $\boldsymbol{C_{min}}$ is the coefficient of the fastest growing term in $\boldsymbol{T(n)}$.
3. $C = \Delta + \frac{3}{2} = 1 + \frac{3}{2} = \frac{5}{2}$
4. Find n_0.
   4.1. Solve $|T(n_0)| \le C|f(n_0)|$

$$\frac{3}{2}n_0^2 + \frac{5}{2}n_0 + 10 \le \frac{5}{2}n_0^2$$

$$n_0^2 - \frac{5}{2}n_0 - 10 \ge 0$$

$$n_0 = \left\lceil \frac{\frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 + 4\cdot 1\cdot 10}}{2} \right\rceil, n_0 > 0$$

$$n_0 = \lceil 4.65 \rceil, n_0 > 0$$

$$n_0 = 5$$

   4.2. Choose an integer value for n_0. Let $n_0 = 5$.

We have shown that $\boldsymbol{T(n) = \frac{3}{2}n^2 + \frac{5}{2}n + 10}$ is $\boldsymbol{O(n^2)}$ because we have found witnesses $\boldsymbol{C} = \frac{5}{2}$ and $n_0 = 5$.

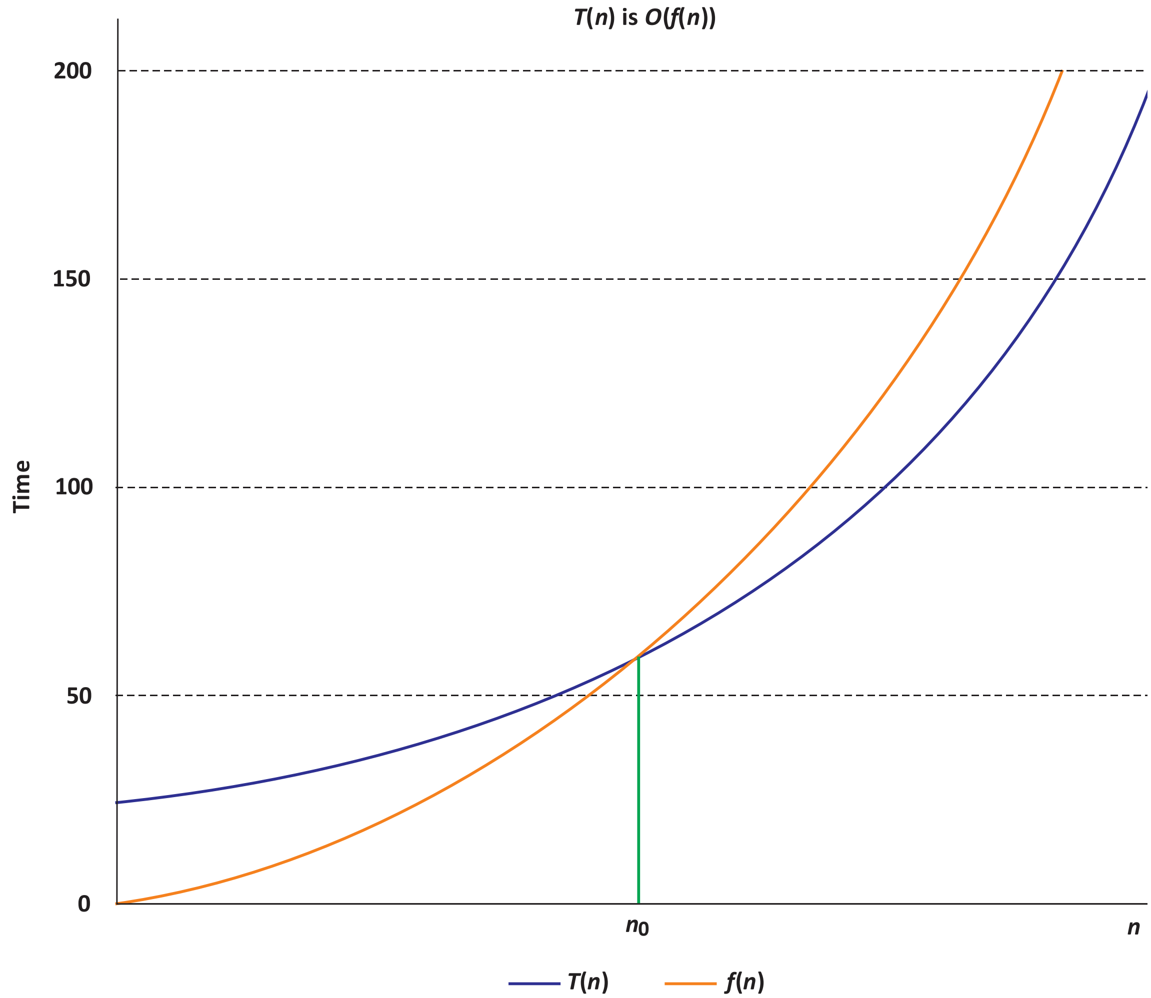

Figure 1. $T(n)$ **is** $O(f(n))$

REMARK	As an estimate of the growth of $T(n)$, $O(f(n))$ is a conservative estimate because it places a ***ceiling*** on the growth $T(n)$. When $n > n_0$, $O(f(n)) \geq T(n)$. Let us call $O(f(n))$, big-O of (n) , the Eeyore estimate because the actual running time of a function, given by $T(n)$ will be ***less*** than the estimate $Cf(n)$ because Eeyore is a pessimist always promising ***less*** than he actually delivers. Seek to emulate Eeyore.

Big-Ω Notation

DEFINITION 2

Let T and f be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $T(n)$ is $\Omega(g(n))$ if there are positive constants n_0 and C such that

$$|T(n)| \geq C|g(n)|$$

whenever

$$n > n_0$$

[This is read as "$T(n)$ is big-Omega of $g(n)$."]

Finding C, n_0, and $f(n)$ for $T(n)$

Steps: Assume $\boldsymbol{T(n) = \frac{3}{2}n^2 + \frac{5}{2}n - 10}$

1. Find $g(n)$. Let $f(n)$ be the fastest growing term in $T(n)$ without its coefficient.
   $\boldsymbol{f(n) = n^2}$
2. Find C.
   2.1. $C = C_{min} - \Delta$, where $\Delta = 1$ (in many cases).
   2.2. $C_{min} = \lim_{n\to\infty} \frac{T(n)}{g(n)} = \lim_{n\to\infty} \frac{\frac{3}{2}n^2 + \frac{5}{2}n - 10}{n^2} = \frac{3}{2}$
   2.3. In practice, C_{min} is the coefficient of the fastest growing term in $T(n)$.
   2.4. $C = \frac{3}{2} - \Delta = \frac{3}{2} - 1 = \frac{1}{2}$
3. Find n_0.
   3.1. Solve $|T(n_0)| \geq C|g(n_0)|$

$$\frac{3}{2}n_0^2 + \frac{5}{2}n_0 - 10 \leq \frac{1}{2}n_0^2$$

$$n_0^2 + \frac{5}{2}n_0 - 10 \leq 0$$

$$n_0 = \left\lfloor \frac{-\frac{5}{2} \mp \sqrt{\left(\frac{5}{2}\right)^2 + 4 \cdot 1 \cdot 10}}{2} \right\rfloor, n_0 > 0$$

$$n_0 = \lfloor 2.15 \rfloor, n_0 > 0$$

$$n_0 \geq 2$$

   3.2. For $n > 0, T(n) \geq Cg(n)$. Select $n_0 = 0$.

We have shown that $\boldsymbol{T(n) = \frac{3}{2}n^2 + \frac{5}{2}n - 10}$ is $\boldsymbol{\Omega(n^2)}$ because we have found witnesses $\boldsymbol{C} = \frac{1}{2}$ and $n_0 = 2$.

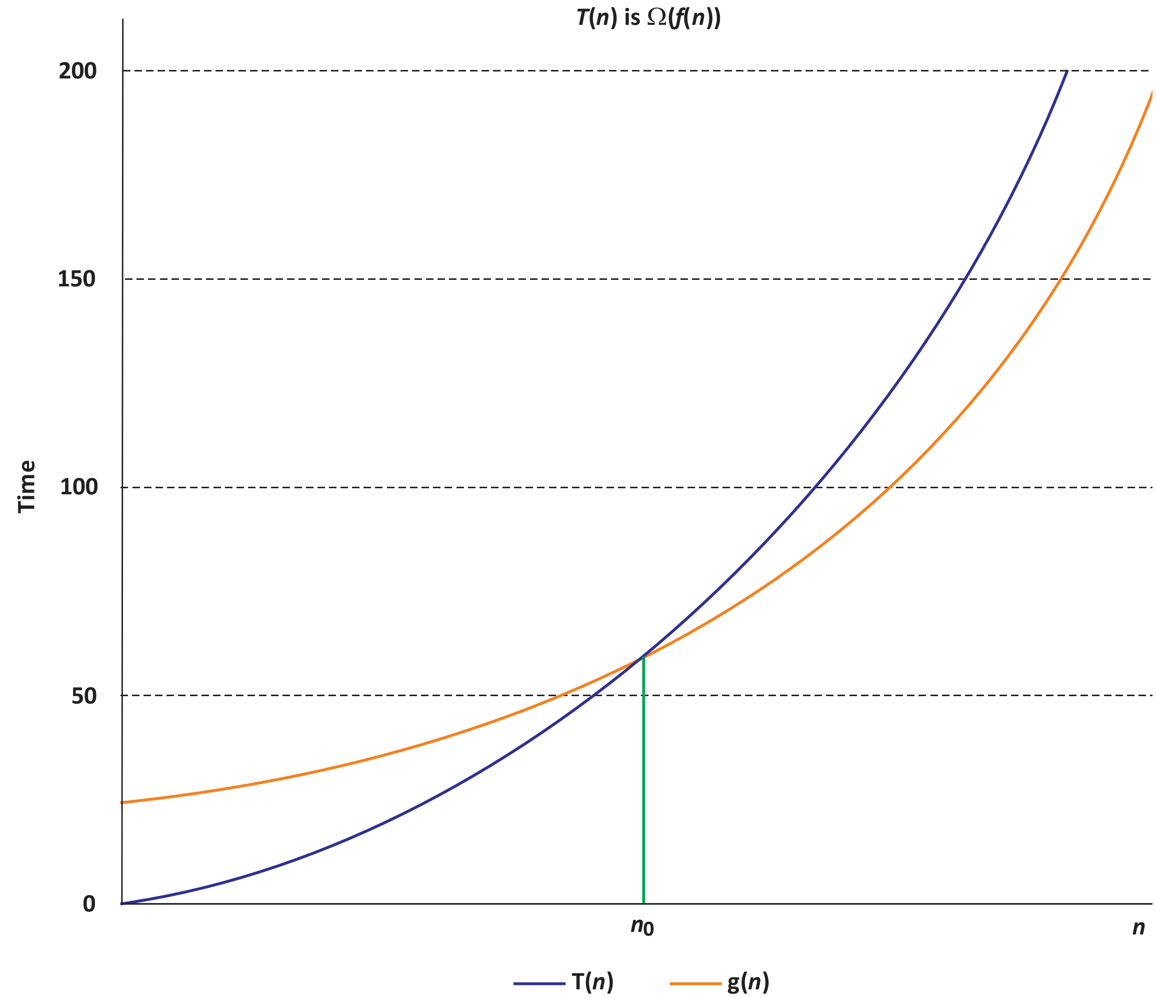

Figure 2. $T(n)$ **is** $\Omega(g(n))$

REMARK

As an estimate of the growth of $T(n)$, $\Omega(g(n))$ is an optimistic estimate because it places a ***floor*** on the growth of $T(n)$. When $n > n_0$, $T(n) \geq \Omega(g(n))$. Let us call $\Omega(g(n))$, big-omega of (n) , the Tigger estimate because the actual running time of a function, given by $T(n)$ will be ***greater*** than the estimate $Cg(n)$ and Tigger is an optimist always promising ***more*** than he can deliver.

Big-Θ Notation

DEFINITION 3 *Let T and f be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $T(n)$ is $\Theta(f(n))$ if $T(n)$ is $O(f(n))$ and if $T(n)$ is $\Omega(f(n))$. When $T(n)$ is $\Theta(f(n))$, we say that"T* is big-Theta of $f(n)$ and we also say that $T(n)$ is of *order* $f(n)$.

Let us find a function, $T(n)$, that is both $O(f(n))$ and $\Omega(f(n))$.

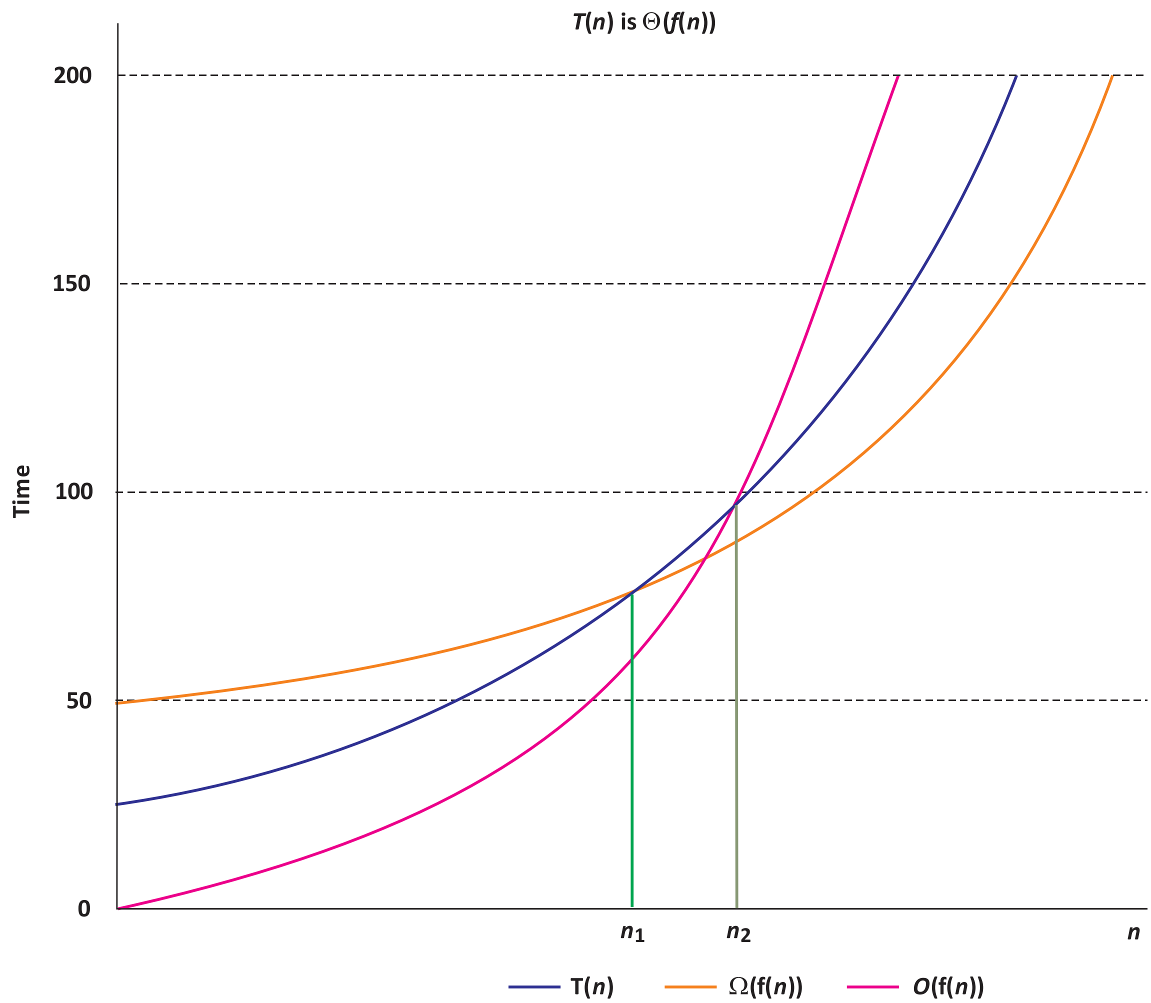

Figure 3. $\boldsymbol{T(n)}$ **is** $\boldsymbol{\Theta(n)}$