Introduction

Good morning. In this section we study sequences. A sequence is an ordered list of elements. Sequences are important to computing because of the iterative nature of computer programs. The time complexity of recursive functions and for-loops are computed by means of sequences and summations.

Sequences

DEFINITION 1 A ***sequence*** is a function from a subset of the set of integers (usually either the set $\{0,1,2\cdots\}$ or the set $\{1,2,3\cdots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

We use the notation $\{a_n\}$ to describe the sequence.

EXAMPLE 1 Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}$$

List the first four terms of the sequence denoted $a_1, a_2,\ a_3,\ a_4$

Solution: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

DEFINITION 2 A ***geometric progression*** is a sequence of the form

$$a, ar, ar^2, \cdots, ar^n, \cdots$$

where the ***initial term a*** and the ***common ratio r*** are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

EXAMPLE 2.1 Consider the sequence $\{b_n\}$, where

$$b_n = (-1)^n$$

List the first four terms of the sequence denoted b_0, b_1, b_2, b_3

Solution: $b_0 = 1, b_1 = -1, b_2 = 1, b_3 = -1$

EXAMPLE 2.2 Consider the sequence $\{c_n\}$, where

$$c_n = 2 \cdot 5^n$$

List the first four terms of the sequence denoted c_0, c_1, c_2, c_3

Solution: $c_0 = 2, c_1 = 10, c_2 = 50, c_3 = 250$

EXAMPLE 2.3 Consider the sequence $\{b_n\}$, where

$$b_n = 6 \cdot \left(\frac{1}{3}\right)^n$$

List the first four terms of the sequence denoted b_0, b_1, b_2, b_3

Solution: $b_0 = 6, b_1 = 2, b_2 = \frac{2}{3}, b_3 = \frac{2}{9}$

DEFINITION 3 An ***arithmetic progression*** is a sequence of the form

$$a, a + d, a + 2d, \cdots, a + nd, \cdots$$

where the ***initial term a*** and the ***common difference d*** are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

EXAMPLE 3.1 Consider the sequence $\{s_n\}$, where

$$s_n = -1 + 4n$$

List the first four terms of the sequence denoted s_0, s_1, s_2, s_3

Solution: $s_0 = -1, s_1 = 3, s_2 = 7, s_3 = 11$

EXAMPLE 3.2 Consider the sequence $\{t_n\}$, where

$$t_n = 7 - 3n$$

List the first four terms of the sequence denoted t_0, t_1, t_2, t_3

Solution: $t_0 = 7, t_1 = 4, t_2 = 1, t_3 = -2$

EXAMPLE 4 The string $abcd$ is a string of length four.

Recurrence Relations

Another way to specify a sequence to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them.

DEFINITION 4 A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$ where n_0 is a nonnegative integer. A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to ***recursively define*** a sequence. We will explain this alternative terminology in Chapter 5.)

EXAMPLE 5 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3, \dots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = a_1 + 3 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$

Excel is an ideal tool for solving these kinds of exercises. Please review the excerpted worksheet below.

i	***a[i]***	**Value**
0	a[0]	2
1	a[1]	5
2	a[2]	8
3	a[3]	11
4	a[4]	14

EXAMPLE 6 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4, \dots$, and suppose that $a_0 = 3$ and $a_1 = 5$ What are a_2, and a_3?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$. It then follows that $a_3 = 2 - 5 = -3$. We can find a_4, a_5, and each successive term in a similar way as shown below.

i	***a[i]***	**Value**
0	a[0]	3
1	a[1]	5
2	a[2]	2
3	a[3]	-3
4	a[4]	-5

DEFINITION 5 The ***Fibonacci sequence***, $f_0, f_1, f_2, \dots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \dots$.

EXAMPLE 7 Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previoius two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

- $f_2 = f_1 + f_0 = 1 + 0 = 1$
- $f_3 = f_2 + f_1 = 1 + 1 = 2$
- $f_4 = f_3 + f_2 = 2 + 1 = 3$
- $f_5 = f_4 + f_3 = 3 + 2 = 5$
- $f_6 = f_5 + f_4 = 5 + 3 = 8$

As in our previous recurrence relations, we can employ Excel to find the values of the desired terms

i	*f[i]*	Value
0	*f[0]*	0
1	*f[1]*	1
2	*f[2]*	1
3	*f[3]*	2
4	*f[4]*	3
5	*f[5]*	5
6	*f[6]*	8
7	*f[7]*	13
8	*f[8]*	21

EXAMPLE 8 Suppose that $\{a_n\}$ is a sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n, where $n = 1, 2, 3, \dots$. Because $n! = n(n-1)(n-2) \dots 2 \cdot 1 = n(n-1)! = na_{n-1}$, we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$.

EXAMPLE 9.1 Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2\big(3(n-1)\big) - 3(n-2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

EXAMPLE 9.2 Determine whether the sequence $\{a_n\}$, where $a_n = 2^n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

Solution: Suppose that $a_n = 2^n$ for every nonnegative integer n. Note that $a_0 = 1, a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

EXAMPLE 9.3 Determine whether the sequence $\{a_n\}$, where $a_n = 5$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$.

Solution: Suppose that $a_n = 5$ for every nonnegative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation.

EXAMPLE 10 Solve the recurrence relation and initial condition in Example 5.

Solution: Recall the recurrence relation in Example 5.

$a_n = a_{n-1} + 3$ for $n = 2,3,4, \ldots$, and suppose that $a_1 = 2$.

We can successively apply the recurrence relation in Example 5, starting with the initial condition $a_1 = 2$, and working upward until we reach a_n to **deduce** a closed formula for the sequence. We see that

- $a_2 = 2 + 3$
- $a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$
- $a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$
- $\vdots$
- $a_n = a_{n-1} + 3 = \left(2 + 3 \cdot (n - 2)\right) + 3 = 2 + 3(n - 1)$

We can also successively apply the recurrence relation in Example 5, starting with the term a_n and working downward until we reach the initial condition $a_1 = 2$ to deduce this same formula. The steps are:

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&= (a_{n-4} + 3) + 3 \cdot 3 = a_{n-4} + 4 \cdot 3 \\
&\vdots \\
&= a_2 + 3(n - 2) = (a_1 + 3) + 3(n - 2) = 2 + 3(n - 1)
\end{aligned}
$$

EXAMPLE 11 **Compound Interest** Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution: To solve this problem, let P_n denote the amount in the account after n years. Because the amount in the account after n years equals the amount in the account after $n-1$ years plus interest for the nth year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10{,}000$.

We can use an iterative approach to find a formula for P_n. Note that

$$\begin{aligned}
P_1 &= (1.11)P_0 \\
P_2 &= (1.11)P_1 = (1.11)^2P_0 \\
P_3 &= (1.11)P_2 = (1.11)^3P_0 \\
&\vdots \\
P_n &= (1.11)P_{n-1} = (1.11)^nP_0
\end{aligned}$$

When we insert the initial condition $P_0 = 10{,}000$, the formula $P_n = (1.11)^nP_0$ is obtained. Inserting $n = 30$ into the formula $P_n = (1.11)^n10{,}000$ shows that after 30 years the account contains

$$P_{30} = (1.11)^{30}10{,}000 = \$228{,}922.97.$$

Special Integer Sequences

Questions for identifying the algebraic representation for a sequence of numeric terms:

- Are there runs of the same value? That is, does the same value occur many times in a row?
- Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence?
- Are terms obtained from previous terms by multiplying by a particular amount?
- Are terms obtained by combining previous terms in a certain way?
- Are there cycles among terms?

EXAMPLE 12.1 Find a formula for the sequence whose first five terms are:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$$

Solution: We recognize an exponential sequence where powers of 2 appear in the denominators 2, 4, 8, and 16. Hence, the sequence is a geometric progression with $a = 1$ and $r = \frac{1}{2}$.

Recall that a geometric progression has the form $a, ar, ar^2, \cdots, ar^n, \cdots$. In this case:

- $a_0 = ar^0 = 1 \cdot \left(\frac{1}{2}\right)^0 = 1$
- $a_1 = ar^1 = 1 \cdot \left(\frac{1}{2}\right)^1 = \frac{1}{2}$
- $a_2 = ar^2 = 1 \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4}$
- $a_3 = ar^3 = 1 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
- $a_4 = ar^4 = 1 \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

EXAMPLE 12.2 Find a formula for the sequence whose first five terms are:

$$1, 3, 5, 7, 9$$

Solution: We recognize a common difference between terms. The difference is two (2). Recalling that the form of an arithmetic progression is $a, a + d, a + 2d, \cdots, a + nd$, we can see that $a = 1$ and $d = 2$.

- $a_0 = a = 1$
- $a_1 = a + d = 1 + 2 = 3$
- $a_2 = a + 2d = 1 + 2 \cdot 2 = 5$
- $a_3 = a + 3d = 1 + 3 \cdot 2 = 7$
- $a_4 = a + 4d = 1 + 4 \cdot 2 = 9$

EXAMPLE 12.3 Find a formula for the sequence whose first five terms are:

$$1, -1, 1, -1, 1$$

Solution: We recognize an exponential sequence where powers of -1 are the terms of the sequence. Odd powers of -1 produce -1 and even powers of -1 produce 1. Hence, the sequence is a geometric progression with $a = 1$ and $r = -1$.

Recall that a geometric progression has the form $a, ar, ar^2, \cdots, ar^n, \cdots$. In this case:

- $a_0 = ar^0 = 1 \cdot (-1)^0 = 1$
- $a_1 = ar^1 = 1 \cdot (-1)^1 = -1$
- $a_2 = ar^2 = 1 \cdot (-1)^2 = 1$
- $a_3 = ar^3 = 1 \cdot (-1)^3 = -1$
- $a_4 = ar^4 = 1 \cdot (-1)^4 = 1$

EXAMPLE 13 How can we produce the terms of a sequence if the first 10 terms are

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4$$

Solution: In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match.

EXAMPLE 14 How can we produce the terms of a sequence if the first 10 terms are

$$5, 11, 17, 23, 29, 35, 41, 47, 53, 59$$

Solution: We need to speculate. Is this an arithmetic progression? Is this a geometric progression? Is this a combination of an arithmetic and geometric progression? Let us guess that the progression is arithmetic and employ the table below to discover the pattern of the sequence.

1. Find the difference between successive terms of the sequence.

i	a_{i+1}	a_i	$a_{i+1} - a_i$
0	11	5	6
1	17	11	6
2	23	17	6
3	29	23	6

We see that this is an arithmetic progression where $d = 6$ and $a = 5$. $a_n = 5 + 6n$, for $n = 0, 1, 2, \dots$.

EXAMPLE 15 How can we produce the terms of a sequence if the first 10 terms are

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123$$

Solution: Observe that each successive term of this sequence, starting with the third term, is the sum of the two previous terms. That is, $4 = 3 + 1, 7 = 4 + 3, 11 = 7 + 4$, and so on. Consequently, if L_n is the nth term in the sequence, we guess that the sequence is determined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_1 = 1$ and $L_2 = 3$ (the same recurrence relation as the Fibonacci sequence, but with different initial conditions). This sequence is known as the **Lucas sequence**, after the French mathematician Francois Édouard Lucas. Lucas studied this sequence and the Fibonacci sequence in the nineteenth century.

EXAMPLE 16 Find a formula for the sequence whose first few terms are:

$$1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047$$

Solution:

1. Find the difference between successive terms of the sequence.

i	a_{i+1}	a_i	$a_{i+1} - a_i$
0	7	1	6
1	25	7	18
2	79	25	54
3	241	79	162

2. Find a common factor in the differences

i	$a_{i+1} - a_i$	Common Factor	Remaining Factor
0	6	2	3
1	18	2	9
2	54	2	27
3	162	2	81

3. Find a pattern in the remaining factors.

i	$a_{i+1} - a_i$	Common Factor	Remaining Factor
0	6	2	$3 = 3^1$
1	18	2	$9 = 3^2$
2	54	2	$27 = 3^3$
3	162	2	$81 = 3^4$

4. Propose a geometric progression

$$a_n = 3^n - 2, n = 1, 2, 3, \cdots$$

5. Test the geometric progression
   - $a_1 = 3^1 - 2 = 1$
   - $a_2 = 3^2 - 2 = 7$
   - $a_3 = 3^3 - 2 = 25$
   - $a_4 = 3^4 - 2 = 79$

Summations

DEFINITION 3.1 Summation notation is used to express the sum of a sequence $\{a_n\}$ and any one of the following forms may be used.

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$

$$\textstyle\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_n$$

or

$$\textstyle\sum_{m \le j \le n} a_j = a_m + a_{m+1} + \cdots + a_n$$

Here, the index of the summation runs through all integers starting with its **lower limit** m and ending with its **upper limit** n. A large uppercase Greek letter sigma, Σ**,** is used to denote summation.

EXAMPLE 17 Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$.

Solution:

$$\sum_{j=1}^{100} \frac{1}{j}$$

EXAMPLE 18 What is the value of $\sum_{j=1}^{5} j^2$?

Solution: We have

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$
$$= 1 + 4 + 9 + 16 + 25$$
$$= 55$$

or from the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{5} k^2 = \frac{5(5+1)(2 \cdot 5 + 1)}{6} = \frac{5 \cdot 6 \cdot 11}{6} = 55$$

EXAMPLE 19 What is the value of $\sum_{k=4}^{8}(-1)^k$?

Solution: We have

$$\sum_{k=4}^{8}(-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$
$$= 1 + (-1) + 1 + (-1) + 1$$
$$= 1$$

EXAMPLE 20 Suppose we have the sum

$$\sum_{j=1}^{5} j^2$$

but want the index of summation to run between 0 and 4 rather than from 1 to 5. To do this, we let $k = j - 1$. Then the new summation index runs from 0 (because $k = 1 - 1 = 0$ when $j = 1$) to 4 (because $k = 5 - 1 = 4$ when $j = 5$), and the term j^2 becomes $(k+1)^2$. Hence,

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4}(k+1)^2$$

It is easily checked that both sums are $1 + 4 + 9 + 16 + 25 = 55$

THEOREM 1 If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

EXAMPLE 21 Evaluate the double sum

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

Solution:

1. First evaluate the inner sum and find a function $f(i)$ where i is the index of the outer sum. In other words, eliminate both the inner summation and the inner index variable.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i) = \sum_{i=1}^{4} 6i$$

2. Next evaluate the outer sum.

$$\sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6 \cdot \frac{4 \cdot 5}{2} = 60$$

TABLE 2 Some Useful Summation Formulae.

Sum	Closed Form
$\sum_{k=0}^{n} ar^k (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, (r \neq 1)$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\left[\frac{n(n+1)}{2}\right]^2$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=0}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

EXAMPLE 23 Evaluate

$$\sum_{k=50}^{100} k^2$$

Solution:

1. $\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$
2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{k=1}^{100} k^2 = \frac{100(101)(201)}{6} = 338{,}350$
4. $\sum_{k=1}^{49} k^2 = \frac{49(50)(99)}{6} = 40{,}425$
5. $\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2 = 338{,}350 - 40{,}425 = 297{,}925$

1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$

Solution:

Part	a_i	Expression	Final Value
a)	a_0	$a_0 = 2 \cdot (-3)^0 + 5^0$	3
b)	a_1	$a_1 = 2 \cdot (-3)^1 + 5^1$	-1
c)	a_4	$a_4 = 2 \cdot (-3)^4 + 5^4$	787
d)	a_5	$a_5 = 2 \cdot (-3)^5 + 5^5$	2,639

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

Solution:

Part	a_n	a_0	a_1	a_2	a_3	a_4
a)	$a_n = 6a_{n-1}, a_0 = 2$	2	12	72	432	2592
		a_1	a_2	a_3	a_4	a_5
b)	$a_n = a_{n-1}^2, a_1 = 2$	2	4	16	256	65536
		a_0	a_1	a_2	a_3	a_4
c)	$a_n = a_{n-1} + 3a_{n-2}$ $a_0 = 1, a_1 = 2$	1	2	5	11	26
d)	$a_n = na_{n-1} + n^2 a_{n-2}$ $a_0 = 1, a_1 = 1$	1	1	6	27	204
e)	$a_n = a_{n-1} + a_{n-3}$ $a_0 = 1, a_1 = 2, a_2 = 0$	1	2	0	1	3

17. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 10.

a) $a_n = 3a_{n-1}, a_0 = 2$

Solution:

$$a_1 = 3a_0 = 3 \cdot 2$$
$$a_2 = 3a_1 = 3 \cdot 3 \cdot 2 = 3^2 \cdot 2$$
$$a_3 = 3a_2 = 3 \cdot 3^2 \cdot 2 = 3^3 \cdot 2$$
$$\vdots$$
$$a_n = 3a_{n-1} = 3 \cdot 3^{n-1} \cdot 2 = 3^n \cdot 2$$

b) $a_n = a_{n-1} + 2, a_0 = 3$

Solution:

$$a_1 = a_0 + 2 = 3 + 2$$
$$a_2 = a_1 + 2 = 3 + 2 + 2 = 3 + 2 \cdot 2$$
$$a_3 = a_2 + 2 = 3 + 2 \cdot 2 + 2 = 3 + 3 \cdot 2$$
$$\vdots$$
$$a_n = a_{n-1} + 2 = 3 + (n-1)2 = 3 + 2n$$

c) $a_n = a_{n-1} + n, a_0 = 1$

Solution:

$a_1 = a_0 + 1 = 1 + 1$
$a_2 = a_1 + 2 = 1 + 1 + 2$
$a_3 = a_2 + 3 = 1 + 1 + 2 + 3$
$a_3 = a_3 + 4 = 1 + 1 + 2 + 3 + 4 = 1 + \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
$\vdots$
$a_n = a_{n-1} + n = 1 + \frac{n(n+1)}{2}$

d) $a_n = a_{n-1} + 2n + 3, a_0 = 4$

Solution:

$a_1 = a_0 + 2 \cdot 1 + 3 = 4 + 2 \cdot 1 + 3$
$a_2 = a_1 + 2 \cdot 2 + 3 = 4 + 2 \cdot 1 + 3 + 2 \cdot 2 + 3 = 4 + 2(1 + 2) + 2 \cdot 3$
$a_3 = a_2 + 2 \cdot 3 + 3 = 4 + 2(1 + 2 + 3) + 3 \cdot 3$
$\vdots$
$a_n = a_{n-1} + 2 \cdot n + 3 = 4 + 2(1 + 2 + \cdots + n) + n \cdot 3 = n^2 + 4n + 4$

e) $a_n = 2a_{n-1} - 1, a_0 = 1$

Solution:

$a_1 = 2a_0 - 1 = 2 \cdot 1 - 1 = 1$
$a_2 = 2a_0 - 1 = 2 \cdot 1 - 1 = 1$
$\vdots$
$a_n = 2a_{n-1} - 1 = 2 \cdot 1 - 1 = 1$

f) $a_n = 3a_{n-1} + 1, a_0 = 1$

Solution:

$a_1 = 3a_0 + 1 = 3 \cdot 1 + 1$
$a_2 = 3a_1 + 1 = 3(3 \cdot 1 + 1) + 1 = 3^2 + 3 + 1$
$a_3 = 3a_2 + 1 = 3(3 \cdot 1 + 1) = 3(3^2 + 3 + 1) + 1 = 3^3 + 3^2 + 3^1 + 3^0$
$\vdots$
$a_n = 3a_n + 1 = 3\sum_{k=0}^{n-1} 3^k + 1 = \sum_{k=0}^{n} 3^k = \frac{3^{n+1} - 1}{2}$

g) $a_n = na_{n-1}, a_0 = 5$

Solution:

$a_1 = 1 \cdot a_0 = 1 \cdot 5$
$a_2 = 2 \cdot a_0 = 2 \cdot 1 \cdot 5$
$a_3 = 3 \cdot a_0 = 3 \cdot 2 \cdot 1 \cdot 5$
$\vdots$
$a_n = n5(n-1)! = 5n!$

h) $a_n = 2na_{n-1}, a_0 = 1$

Solution:

$a_1 = 2 \cdot 1 \cdot a_0 = 2 \cdot 1 \cdot 1$
$a_2 = 2 \cdot 2 \cdot a_1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$
$a_3 = 2 \cdot 3 \cdot a_2 = 2 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 2^3 3!$
$\vdots$
$a_n = 2 \cdot n \cdot a_{n-1} = 2n2^{n-1}(n-1)! = 2^n n!$

19. Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

Solution:

Since the number of bacteria triples every hour, the recurrence relation should say that the number of bacteria after n hours is 3 times the number of bacteria after $n-1$ hours. Letting b_n denote the number of bacteria after n hours, this statement translates into the recurrence relation

$$b_n = 3b_{n-1}$$

b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

Solution:

The given statement is the initial condition $b_0 = 100$ (the number of bacteria at the beginning is the number of bacteria after no hours have elapsed). We solve the recurrence relation by iteration:

$$b_1 = 3b_0$$
$$b_2 = 3b_1 = 3 \cdot 3b_0 = 3^2 b_0$$
$$\vdots$$
$$b_n = 3b_{n-1} = 3 \cdot 3^{n-1} b_0 = 3^n b_0$$

Hence, the number of bacteria in the colony after 10 hours is $b_{10} = 3^{10} b_0 = 3^{10} \cdot 100 = 5,904,900$