**Introduction**

**DEFINITION 1**  A set is an unordered collection of objects.

**DEFINITION 2**  The objects in a set are called the *elements*, or *members*, of the set. A set is said to *contain* its elements.

**EXAMPLE 1** Describe the vowels in the English alphabet by employing the language of sets.  
*Solution*: The set $V$ of all vowels in the English alphabet can be written as $V = \{a,e,i,o,u\}$.

**EXAMPLE 2** Describe the collection of odd positive integers whose values are smaller than ten by employing the language of sets.  
*Solution*: Let $O$ be the set of odd positive integers less than 10. $O = \{1,3,5,7,9\}$.

**EXAMPLE 3** Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a,2,Fred,New\text{ }Jersey\}$ is the set containing the four elements $a,2,Fred,$ and $New\text{ }Jersey$.

**Remark**  Sets contain unique elements. For example, the set of vowels in the English language is seldom expressed as $V = \{a,a,e,e,i,i,o,o,u,u\}$. An element should only appear only once.

**EXAMPLE 4** Describe the collection of integers whose values are between than one and 99 by employing an ellipsis and set notation.  
*Solution*: $I = \{1,2,3,\ldots,99\}$.

**Set builder notation**  We characterize all those elements in the set by stating the property or properties they must have to be members.

**Set builder notation**  
- $O = \{x|x \text{ is odd positive integer less than } 10\}$.  
- $O = \{x \in \mathbb{Z}^+ | x \text{ is odd and } x < 10\}$.  
- $\mathbb{Q}^+ = \{x \in \mathbb{R} | x = \frac{p}{q}, p \in \mathbb{Z}^+, q \in \mathbb{Z}^+\}$

**Explanation**  
- $O = \{1,3,5,7,9\}$.  
- $O$ is the set of positive integers given that $x$ is odd and $x$ is less than 10.  
- $\mathbb{Q}^+$ is the set of all positive rational numbers.

**Common Sets**  
- $\mathbb{N} = \{0,1,2,\ldots\}$, the set of *natural numbers*  
- $\mathbb{Z} = \{\ldots,-2,-1,0,1,2,\ldots\}$, the set of *integers*  
- $\mathbb{Z}^+ = \{1,2,3,\ldots\}$, the set of *positive integers*  
- $\mathbb{Q} = \{\frac{p}{q} | p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$, the set of *rational numbers*  
- $\mathbb{R}$ the set of *real numbers*
EXAMPLE 5  The set \( \{ \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \} \) is a set containing four elements, each of which is a set. The four elements of this set are \( \mathbb{N} \), the set of natural numbers; \( \mathbb{Z} \), the set of integers; \( \mathbb{Q} \), the set of rational numbers; and \( \mathbb{R} \), the set of real numbers.

Remark  Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *Boolean* is the name of the set \( \{0,1\} \) together with operators on one or more elements of this set, such as AND, OR, and NOT.

DEFINITION 3  Two sets are equal if and only if they have the same elements. That is if \( A \) and \( B \) are sets, then \( A \) and \( B \) are equal if and only if \( \forall x (x \in A \iff x \in B) \). We write \( A = B \) if \( A \) and \( B \) are equal sets.

EXAMPLE 6  The sets \( \{1,3,5\} \) and \( \{3,5,1\} \) are equal, because they have the same elements. (Recall that a set is a collection of unordered elements from Definition 1.)

1. Note that the order in which the elements of a set are listed does not matter.
2. Note also that it does not matter if an element of a set is listed more than once, so \( \{1,3,3,3,5,5,5\} \) is the same as the set \( \{1,3,5\} \) because they have the same elements.

EXAMPLE 7  Draw a Venn diagram that represents \( V \), the set of vowels in the English alphabet.

Solution: 

![Venn Diagram](image_url)
DEFINITION 4

The set $A$ is said to be a subset of $B$ if and only if every element of $A$ is also an element of $B$. We use the notation $A \subseteq B$ to indicate that $A$ is a subset of the set $B$.

EXAMPLE 8

The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself).

THEOREM 1

For every set $S$,

(i) $\emptyset \subseteq S$ and
(ii) $S \subseteq S$

DEFINITION 4.1

Set $A$ is a **proper subset** of $B$ when $A \subseteq B$ and there is at least one element in $B$ that is not in $A$. We use the notation $A \subset B$ when $A$ is a proper subset of $B$.

EXAMPLE 8.1

Consider the two sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3,4\}$. $B$ is a proper subset of $A$ and we write $B \subset A$.

DEFINITION 5

Let $S$ be a set. If there are exactly $n$ distinct elements in $S$ where $n$ is a nonnegative integer, we say that $S$ is a finite set and that $n$ is the **cardinality** of $S$. The cardinality of $S$ is also denoted by $|S|$.

EXAMPLE 9

Let $A$ be the set of odd positive integers less than 10. $A = \{1,3,5,7,9\}$ and $|A| = 5$.

EXAMPLE 10

Let $S$ be the set of letters in the English alphabet. Then $|S| = 26$.

EXAMPLE 11

Because the null set has no elements, it follows that $|\emptyset| = 0$.

DEFINITION 6

A set is said to be infinite if it is not finite.

EXAMPLE 12

The set of positive integers is infinite.
**The Power Set**

**DEFINITION 7** Given a set $S$, the power set of $S$ is the set of all subsets of the set $S$. The power set of $S$ is denoted by $P(S)$.

**EXAMPLE 13** What is the power set of $\{0,1,2\}$?

*Solution* $P(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

**EXAMPLE 14** What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

*Solution* The empty set has exactly one subset, namely, itself. Consequently, $P(\emptyset) = \{\emptyset\}$

**Remark** If a set has $n$ elements, then its power set has $2^n$ elements.

**Cartesian Product**

**DEFINITION 8** The ordered $n$-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has $a_1$ as its first element, $a_2$ as its second element, ..., and $a_n$ as its $n$th element.

**DEFINITION 9** Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) | a \in A, b \in B\}$.

**EXAMPLE 15** Let $A$ represent the set of all students at a university, and let $B$ represent the set of all courses offered at the university. What is the Cartesian product $A \times B$?

*Solution* The Cartesian product $A \times B$ consists of all the ordered pairs of the form $(a, b)$, where $a$ is a student and $b$ is a course offered at the university. The set $A \times B$ can be used to represent all possible enrollments of students in courses at the university.

**EXAMPLE 16** What is the Cartesian product of $A = \{1,2\}$ and $B = \{a, b, c\}$?

*Solution* The Cartesian product $A \times B$ is $A \times B = \{(1, a), (1, b)(1, c), (2, a), (2, b), (2, c)\}$.

**Remark** relation

**EXAMPLE 17** under construction

*Solution*

**DEFINITION 10** under construction

**EXAMPLE 18** under construction

*Solution*
Using Set Notation with Quantifiers

EXAMPLE 19  under construction  Solution

Truth Sets of Quantifiers

EXAMPLE 19  under construction  Solution
2. Use set builder notation to give a description of each of these sets.

Solution:

<table>
<thead>
<tr>
<th>Part</th>
<th>Set</th>
<th>Set Builder Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>{0, 5, 10, 15, 20}</td>
<td>{5n</td>
</tr>
<tr>
<td>b)</td>
<td>{-4, -2, 0, 2, 4}</td>
<td>{2x</td>
</tr>
<tr>
<td>c)</td>
<td>{a, b, c}</td>
<td>{x</td>
</tr>
</tbody>
</table>

3. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

Solution:

<table>
<thead>
<tr>
<th>Part</th>
<th>(A)</th>
<th>(B)</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>the set of airline flights from New York to New Delhi</td>
<td>The set of nonstop airline flights from New York to New Delhi</td>
<td>(B \subseteq A)</td>
</tr>
<tr>
<td>b)</td>
<td>the set of people who speak English</td>
<td>The set of people who speak Chinese</td>
<td>Neither</td>
</tr>
<tr>
<td>c)</td>
<td>The set of flying squirrels</td>
<td>The set of living creatures that can fly.</td>
<td>(A \subseteq B)</td>
</tr>
</tbody>
</table>

16. Use a Venn diagram to illustrate the relationships \(A \subset B\) and \(B \subset C\).

Solution: