Introduction

DEFINITION 1 A *set* is an unordered collection of objects.

DEFINITION 2 The objects in a set are called the *elements*, or *members*, of the set. A set is said to *contain* its elements.

EXAMPLE 1 Describe the vowels in the English alphabet by employing the language of sets.
Solution: The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

EXAMPLE 2 Describe the collection of odd positive integers whose values are smaller than ten by employing the language of sets.
Solution: Let O be the set of odd positive integers less than 10. $O = \{1,3,5,7,9\}$.

EXAMPLE 3 Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a, 2, Fred, New\ Jersey\}$ is the set containing the four elements a, 2, $Fred$, and $New\ Jersey$.

Remark Sets contain unique elements. For example, the set of vowels in the English language is seldom expressed as $V = \{a, a, e, e, i, i, o, o\ , u, u\}$. An element should only appear only once.

EXAMPLE 4 Describe the collection of integers whose values are between than one and 99 by employing an ellipsis and set notation.
Solution: $I = \{1,2,3, \cdots, 99\}$.

Set builder notation We characterize all those elements in the set by stating the property or properties they must have to be members.

Set builder notation	**Explanation**
$O = \{x \mid x$ is odd positive integer less than 10$\}$. $O = \{x \in \mathbb{Z}^+ \mid x\ is\ odd\ and\ x < 10\}$.	$O = \{1,3,5,7,9\}$. O is the set of positive integers given that x is odd and x is less than 10.
$\mathbb{Q}^+ = \{x \in \mathbb{R} \mid x = {}^{p}/_{q}, p \in \mathbb{Z}^+, q \in \mathbb{Z}^+\}$	$\mathbb{Q}^+$ is the set of all positive rational numbers.

Common Sets	**Description**
$\mathbb{N}$	$\mathbb{N} = \{0,1,2, \cdots\}$, the set of **natural numbers**
$\mathbb{Z}$	$\mathbb{Z} = \{\cdots, -2, -1,0,1,2, \cdots\}$, the set of **integers**
$\mathbb{Z}^+$	$\mathbb{Z}^+ = \{1,2,3, \cdots\}$, the set of **positive integers**
$\mathbb{Q}$	$\mathbb{Q} = \{{}^{p}/_{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$, the set of **rational numbers**
$\mathbb{R}$	the set of **real numbers**

EXAMPLE 5 The set $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are $\mathbb{N}$, the set of natural numbers; $\mathbb{Z}$, the set of integers; $\mathbb{Q}$, the set of rational numbers; and $\mathbb{R}$, the set of real numbers.

Remark Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *Boolean* is the name of the set $\{0,1\}$ together with operators on one or more elements of this set, such as AND, OR, and NOT.

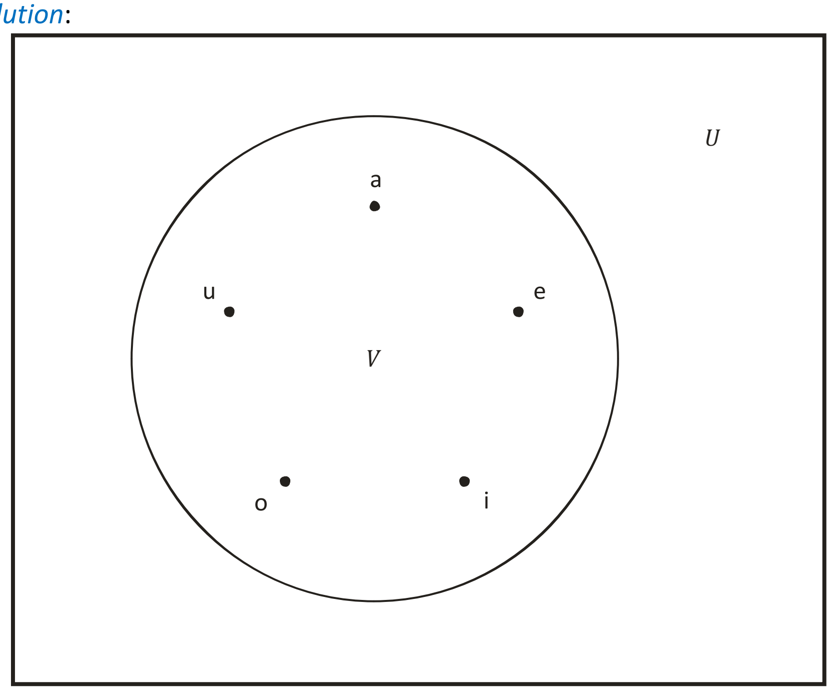
DEFINITION 3 Two sets are equal if and only if they have the same elements. That is if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

EXAMPLE 6 The sets $\{1,3,5\}$ and $\{3,5,1\}$ are equal, because they have the same elements. (Recall that a set is a collection of unordered elements from Definition 1.}

1. Note that the order in which the elements of a set are listed does not matter.
2. Note also that it does not matter if an element of a set is listed more than once, so $\{1,3,3,3,5,5,5,5\}$ is the same as the set $\{1,3,5\}$ because they have the same elements.

EXAMPLE 7 Draw a Venn diagram that represents V, the set of vowels in the English alphabet.

Solution:



DEFINITION 4 The set A is said to be a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

EXAMPLE 8 The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself).

THEOREM 1 For every set S,

(i) $\emptyset \subseteq S$ and

(ii) $S \subseteq S$

DEFINITION 4.1 Set A is a **proper subset** of B when $A \subseteq B$ and there is at least one element in B that is not in A. We use the notation $A \subset B$ when A is a proper subset of B.

EXAMPLE 8.1 Consider the two sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3,4\}$. B is a proper subset of A and we write $B \subset A$.

DEFINITION 5 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the ***cardinality*** of S. The cardinality of S is also denoted by $|S|$.

EXAMPLE 9 Let A be the set of odd positive integers less than 10. $A = \{1,3,5,7,9\}$ and $|A| = 5$.

EXAMPLE 10 Let S be the set of letters in the English alphabet. Then $|S| = 26$.

EXAMPLE 11 Because the null set has no elements, it follows that $|\emptyset| = 0$.

DEFINITION 6 A set is said to be infinite if it is not finite.

EXAMPLE 12 The set of positive integers is infinite.

The Power Set

DEFINITION 7 Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.

EXAMPLE 13 What is the power set of $\{0,1,2\}$?
Solution $P(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

EXAMPLE 14 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?
Solution The empty set has exactly one subset, namely, itself. Consequently,

$$P(\{\emptyset\}) = \{\emptyset\}$$

Remark If a set has n elements, then its power set has $\mathbf{2^n}$ elements.

Cartesian Product

DEFINITION 8 The ordered n-tuple $(a_1, a_2, \cdots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its nth element.

DEFINITION 9 Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) | a \in A, b \in B\}$.

EXAMPLE 15 Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$?
Solution The Cartesian production $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student and b is a course offered at the university. The set $A \times B$ can be used to represent all possible enrollments of students in courses at the university.

EXAMPLE 16 What is the Cartesian product of $A = \{1,2\}$ and $B = \{a, b, c\}$?
Solution The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b)(1, c), (2, a), (2, b), (2, c)\}.$$

Remark **relation**

EXAMPLE 17 under construction
Solution

DEFINITION 10 under construction

EXAMPLE 18 under construction
Solution

Using Set Notation with Quantifiers

EXAMPLE 19 under construction

Solution

Truth Sets of Quantifiers

EXAMPLE 19 under construction

Solution

2. Use set builder notation to give a description of each of these sets.
Solution:

Part	Set	Set Builder Notation
a)	$\{0, 5, 10, 15, 20\}$	$\{5n \mid n = 0, 1, 2, 3, 4\}$
b)	$\{-4, -2, 0, 2, 4\}$	$\{2x \mid x \in \mathbb{Z}, -2 \le x \le 2\}$
c)	$\{a, b, c\}$	$\{x \mid x \in \textit{lower case English alphabet}, x \le c\}$

3. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.
Solution:

Part	A	B	Relationship
a)	the set of airline flights from New York to New Delhi	The set of nonstop airline flights from New York to New Delhi	$B \subseteq A$
b)	the set of people who speak English	The set of people who speak Chinese	Neither
c)	The set of flying squirrels	The set of living creatures that can fly.	$A \subseteq B$

16. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.
Solution: