## Introduction

In this section we will introduce a more powerful type of logic called **predicate logic**.

## Predicates

Consider the statement: $x > 3$. The statement has two parts:

1. the variable, $x$ and
2. the **predicate**, is greater than 3, $> 3$.

The statement "$x$ is greater than 3", $x > 3$, can be denoted by $P(x)$. The statement $P(x)$ is said to be the value of the **propositional function** $P$ at $x$. Once a value has been assigned to the variable $x$, the statement $P(x)$ becomes a proposition and has a truth value.

**EXAMPLE 1** Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

| Proposition | Application | Truth Value |
|---|---|---|
| $P(4)$ | $\mathbf{4} > 3$ | **True** |
| $P(2)$ | $\mathbf{2} > 3$ | **False** |

**EXAMPLE 2** Let $A(x)$ denote the statement "Computer $x$ is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are the truth values of $A(CS1)$, $A(CS2)$, and $A(MATH1)$.

| Proposition | Application | Truth Value |
|---|---|---|
| $A(CS1)$ | Computer *CS1* is under attack by an intruder. | **False** |
| $A(CS2)$ | Computer *CS2* is under attack by an intruder. | **True** |
| $A(MATH1)$ | Computer *MATH1* is under attack by an intruder. | **True** |

**EXAMPLE 3** Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1,2)$ and $Q(3,0)$?

| Proposition | Application | Truth Value |
|---|---|---|
| $Q(1,2)$ | $1 = 2 + 3$ | **False** |
| $Q(3,0)$ | $3 = 0 + 3$ | **True** |

In general, a statement involving the $n$ variables $x_1, x_2, \cdots, x_n$ can be denoted by

$$P(x_1, x_2, \cdots, x_n) .$$

A statement of the form $P(x_1, x_2, \cdots, x_n)$ is the value of the **propositional function** $P$ at the $n$-tuple $(x_1, x_2, \cdots, x_n)$, and $P$ is also called an $n$-**place predicate** or a $n$-**ary predicate**.

**EXAMPLE 6** Consider the statement

**if** $x > 0$ **then** $x := x + 1$.

When this statement is encountered in a program, the value of the variable $x$ at that point in the execution of the program is inserted into $P(x)$, which is "$x > 0$." If $P(x)$ is true for this value of $x$, the assignment statement $x := x + 1$ is executed, so the value of $x$ is increased by 1. If $P(x)$ is false for this value of $x$, the assignment statement is not executed, so the value of $x$ is not changed.

## Quantifiers

**DEFINITION 1** *The universal quantification of $P(x)$ is the statement*

"$P(x)$ for all values of $x$ in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the ***universal quantifier***. We read $\forall x P(x)$ as "for all $x$ $P(x)$" or "for every $x$ $P(x)$." An element for which $P(x)$ is false is called a **counter example** of $\forall x P(x)$.

| **TABLE 1 Quantifiers** | | |
|---|---|---|
| ***Statement*** | ***When True?*** | ***When False?*** |
| $\forall x P(x)$ | $P(x)$ is true for every $x$. | There is an $x$ for which $P(x)$ is false. |
| $\exists x P(x)$ | There is an $x$ for which $P(x)$ is true. | $P(x)$ is false for every $x$. |

**EXAMPLE 8** Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers.

*Solution*: Because $P(x)$ is true for all real numbers $x$, the quantification

$$\forall x P(x)$$

is true.

**EXAMPLE 9** Let $Q(x)$ be the statement "$x > 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

*Solution*: $Q(x)$ is not true for every real number $x$, because, **for instance, $Q(1)$ is false**. That is, $x = 1$ is a counter example for the statement $\forall x Q(x)$. Thus,

$$\forall x Q(x)$$

is false.

When all the elements in the domain can be listed – say, $x_1, x_2, \cdots, x_n$ - it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

because this conjunction is true if and only if $P(x_1), P(x_2), \cdots, P(x_n)$ are all true.

**EXAMPLE 11** What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

*Solution*: The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$" is false, it follows that $\forall x P(x)$ is false.

**DEFINITION 2** *The existential quantification of $P(x)$ is the proposition*

"There exists an element $x$ in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for all the existential quantification of $P(x)$. Here $\exists$ is called the ***existential quantifier***.

- A **domain must always** be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning.
- Besides the words "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is." The existential quantification $\exists x P(x)$ is read as

"There is an $x$ such that $P(x)$,"
"There is at least one $x$ such that $P(x)$,"

or

"For some $x$ $P(x)$."

**EXAMPLE 14** Let $P(x)$ be the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

*Solution*: Because "$x > 3$" is sometimes true – for instance, when $x = 4$ $P(x)$ - the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

**EXAMPLE 15** Let $Q(x)$ be the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

*Solution*: Because $Q(x)$ is false for every real number $x$, the existential quantification of $Q(x)$, which is $\exists x Q(x)$ is false.

***Remark***: Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty. If the domain is empty, then $\exists x Q(x)$ is false whenever $Q(x)$ is a propositional function because when the domain is empty, there can be no element $x$ in the domain for which $Q(x)$ is true.

When all the elements in the domain can be listed – say, $x_1, x_2, \cdots, x_n$ - it follows that the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \cdots, P(x_n)$ is true.

**EXAMPLE 16** What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

*Solution*: Because the domain is $\{1,2,3,4\}$, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true.

## Other Quantifiers

| Name | Notation | Description |
|---|---|---|
| **uniqueness quantifier** | ∃! | The notation $\exists!\, x P(x)$ states "There exists a unique $x$ such that $P(x)$ is true." Other phrases for uniqueness quantification include "there is exactly one" and "there is one and only one." |
| **uniqueness quantifier** | $\exists_1$ | $\exists_1 x P(x)$ |

## Quantifiers with Restricted Domains

- A condition appears after the quantifier. The condition is an expression that a variable must satisfy.

**EXAMPLE 17** What do the statements $\forall x < 0(x^2 > 0), \forall y \neq 0(y^3 \neq 0)$, and $\exists z > 0(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

*Solution*: The statement $\forall x < 0(x^2 > 0)$ states that for every real number $x$ with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x < 0 \rightarrow x^2 > 0)$.

The statement $\forall y \neq 0(y^3 \neq 0)$ states that for every real number $y$ with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." Note that this statement is equivalent to , $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.

Finally, the statement $\exists z > 0(z^2 = 2)$ states that there exists a real number $z$ with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$.

## Precedence of Quantifiers

The quantifiers ∀ and ∃ have higher precedence than all logical operators from propositional calculus. For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$. In other words, it means $(\forall x P(x)\ ) \vee (Q(x))$ rather than $\forall x(P(x) \vee Q(x))$.

## Binding Variables

- An occurrence of a variable may be **bound**.
  - An occurrence of a variable may be bound by a quantifier. For example $\forall x P(x, y)$ makes variable $x$ bound and variable $y$ free.
  - An occurrence of a variable may be bound by being set to a particular value.
- An occurrence of a variable may be **free**.

**EXAMPLE 18** In the statement $\exists x(x + y = 1)$, the variable $\boldsymbol{x}$ **is bound** by the existential quantification $\exists x$, but the variable $\boldsymbol{y}$ **is free** because it is not bound by a quantifier and no value is assigned to this variable.

## Logical Equivalences Involving Quantifiers

**DEFINITION 3** Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements $S$ and $T$ involving predicates and quantifiers are logically equivalent.

**EXAMPLE 19** Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent (where the same domain is used throughout).

*Solution*: To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates $P$ and $Q$ are, and no matter which domain of discourse is used. We can show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent by doing two things. First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall x P(x) \wedge \forall x Q(x)$ is true. Second, we show that if $\forall x P(x) \wedge \forall x Q(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

Assume $\forall x(P(x) \wedge Q(x))$ is true.

1. Let $a$ be an element in the domain of discourse.
2. $P(a) \wedge Q(a)$ is true by employing the assumption.
3. $P(a)$ and $Q(a)$ is true by employing the definition of $\wedge$.
4. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain we can conclude that $\forall x P(x)$ and $\forall x Q(x)$ are both true.
5. This means that $\forall x P(x) \wedge \forall x Q(x)$

Assume $\forall x P(x) \wedge \forall x Q(x)$ is true.

1. $\forall x P(x)$ is true.
2. $\forall x Q(x)$ is true.
3. Let $a$ be an element in the domain of discourse.
4. $P(a)$ is true because $\forall x P(x)$ is true.
5. $Q(a)$ is true because $\forall x Q(x)$ is true.
6. For all $a$, $P(a) \wedge Q(a)$ is true.
7. It follows that $\forall x(P(x) \wedge Q(x))$.

We conclude $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$

## Negating Quantified Expressions

Consider the statement
"Every student in your class has taken a course in calculus."
This statement is a universal quantification, namely,

$$\forall x P(x)$$

where $P(x)$ is the statement
"$x$ has taken a course in calculus."

The negation of the statement is
"It is not the case that every student in your class has taken a course in calculus."
This is equivalent to,
"There is a student in your class who has not taken a course in calculus."

The foregoing example illustrates the following logical equivalence.

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

| **TABLE 2 De Morgan's Laws for Quantifiers** | | | |
|---|---|---|---|
| ***Negation*** | ***Equivalent Statement*** | ***When Is Negation True?*** | ***When False?*** |
| $\neg\exists x P(x)$ | $\forall x \neg P(x)$ | For every $x$, $P(x)$ is false | There is an $x$, for which $P(x)$ is true. |
| $\neg\forall x P(x)$ | $\exists x \neg P(x)$ | There is an $x$, for which $P(x)$ is false. | $P(x)$ is true for every $x$. |

***Remark***: When the domain of a predicate $P(x)$ consists of $n$ elements, where $n$ is a positive integer, the rules for negating quantified statements are exactly the same as De Morgan's laws discussed in Section 1.2. This is why these rules are called De Morgan's laws for quantifiers. When the domain has $n$ elements $x_1, x_2, \cdots, x_n$, it follows that $\neg\forall x P(x)$ is the same as $\neg\big(P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)\big)$, which is equivalent to $\neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n)$ by De Morgan's laws, and this is the same as $\exists x \neg P(x)$. Similarly, $\neg\exists x P(x)$ is the same as $\neg(\, P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n))$, which by De Morgan's laws is equivalent to $\neg P(x_1) \wedge \neg P(x_2) \wedge \cdots \wedge \neg P(x_n)$, and this is the same as $\forall x \neg P(x)$.

**EXAMPLE 20.1** What is the negation of the statement "There is an honest politician?"

*Solution*: Let $H(x)$ denote "$x$ is honest." Then the statement "There is an honest politician" is represented by $\exists x H(x)$, where the domain consists of all politicians.

The negation of this statement is $\neg\exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This negation can be expressed unambiguously in English as "Every politician is dishonest."

**EXAMPLE 21.1** What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

*Solution*: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x \neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The truth values of these statements depend on the domain.

**EXAMPLE 22** Show that $\neg\forall x(P(x) \to Q(x)) \equiv \exists x(P(x) \land \neg Q(x))$

*Solution*:

| Expression | Justification |
|---|---|
| $\neg\forall x(P(x) \to Q(x))$ | Initial assumption |
| $\exists x(\neg(P(x) \to Q(x)))$ | Table 2, row 2 |
| $\exists x(P(x) \land \neg Q(x))$ | Section 1.2, Table 7, row 5 |
| **Expression** | **Justification** |
| $\exists x(P(x) \land \neg Q(x))$ | Initial assumption |
| $\exists x(\neg(P(x) \to Q(x)))$ | Section 1.2, Table 7, row 5 |
| $\neg\forall x(P(x) \to Q(x))$ | Table 2, row 2 |

## Translating from English into Logical Expressions

**EXAMPLE 23** Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

*Solution*: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use.

"For every student in this class, that student has studied calculus."

Next, we introduce a variable $x$ so that our statement becomes

"For every student $x$ in this class, $x$ has studied calculus."

We introduce $C(x)$, which is the statement "$x$ has studied calculus." We also confine the domain of discourse to students in the class. Our statement can be expressed

$$\forall x C(x)$$

If we wish to change to domain of discourse to consist of all people, we need to express our statement as

"For every person $x$, if the person $x$ is a student in this class then $x$ has studied calculus."

Let $S(x)$ represent the statement "$x$ is a person in this class." Our statement can be represented as

$$\forall x(S(x) \to C(x))$$

*Caution!* Our statement *cannot* be expressed as $\forall x(S(x) \land C(x))$ because this statement says that all people are students in this class and have studied calculus!

## Using Quantifiers in System Specifications

- under construction

## Examples from Lewis Carroll

- under construction

## Logic Programming

- under construction

3. Let $Q(x, y)$ denote the statement "the word $x$ is the capital of $y$." What are these truth values?
**Solution:**

| Part | Predicate | Truth Value |
|---|---|---|
| a | $Q(\text{Denver}, \text{Colorado})$ | T |
| b | $Q(\text{Detroit}, \text{Michigan})$ | F |
| c | $Q(\text{Massachusetts}, \text{Boston})$ | F |
| d | $Q(\text{New York}, \text{New York})$ | F |

7. Translate these statements into English where $C(x)$ is "$x$ is a comedian" and $F(x)$ is "$x$ is funny" and the domain consists of all people.
**Solution:**

| Part | Logical Expression | English Equivalent |
|---|---|---|
| a | $\forall x(C(x) \rightarrow F(x))$ | For every $x$, if $x$ is a comedian, then $x$ is funny. |
| | | Every comedian is funny. |
| b | $\forall x(C(x) \wedge F(x))$ | For every $x$, $x$ is a comedian *and* $x$ is funny. |
| | | Every person is a funny comedian. |
| c | $\exists x(C(x) \rightarrow F(x))$ | There exists an $x$, if $x$ is a comedian, then $x$ is funny. |
| | | There exists a person such that if he or she is a comedian, then he or she is funny. |
| d | $\exists x(C(x) \wedge F(x))$ | There exists an $x$, $x$ is a comedian *and* $x$ is funny. |
| | | There exists a funny comedian |

13. Determine the truth value of each of these statements if the domain consists of all integers
**Solution:**

| Part | Predicate | Justification | Truth Value |
|---|---|---|---|
| a | $\forall n(n + 1 > n)$ | Subtracting $n$ from both sides of the inequality we find $$1 > 0$$ which we know is always true. | T |
| b | $\exists n(2n = 3n)$ | We need only find an integer that makes the predicate true for the statement to be true. Such an integer is zero (0). $$2 \cdot 0 = 3 \cdot 0 = 0$$ | T |
| c | $\exists n(n = -n)$ | We need only find an integer that makes the predicate true for the statement to be true. Such an integer is zero (0). $$0 = -0 = 0$$ | T |
| d | $\forall n(3n \leq 4n)$ | We need only find an integer that makes the predicate false for the statement to be false. The negative integers make the predicate false. $$3(-1) \nleq 4(-1)$$ | F |

27. Translate these statements into logical expressions in three different ways by varying the domain and by using predicates with one and two variables.
    Solution:
    a) A student in your school has lived in Vietnam.
    **Solution:**

| Predicate | Meaning |
|---|---|
| $Y(x)$ | $x$ is in your school. |
| $V(x)$ | $x$ has lived in Vietnam. |
| $D(x,y)$ | person $x$ has lived country $y$. |

| | | |
|---|---|---|
| domain | the domain is just your schoolmates | $\exists x V(x)$ |
| one variable | the domain is all people | $\exists x(Y(x) \land V(x))$ |
| two variables | the domain is all people | $\exists x(Y(x) \land D(x, Vietnam))$ |

    b) There is a student in your school who cannot speak Hindi.
    **Solution:**

| Predicate | Meaning |
|---|---|
| $Y(x)$ | $x$ is in your class. |
| $H(x)$ | $x$ can speak Hindi. |
| $S(x,y)$ | person $x$ can speak language $y$. |

| | | |
|---|---|---|
| domain | the domain is just your class | $\exists x(\neg H(x))$ |
| one variable | the domain is all people | $\exists x(Y(x) \land \neg H(x))$ |
| two variables | the domain is all people | $\exists x(Y(x) \land \neg S(x, \text{Hindi}))$ |

    c) A student in your school knows Java, Prolog, and, C++.
    **Solution:**

| Predicate | Meaning |
|---|---|
| $Y(x)$ | $x$ is in your school. |
| $J(x)$ | $x$ knows Java. |
| $P(x)$ | $x$ knows Prolog. |
| $C(x)$ | $x$ knows C++. |
| $K(x,y)$ | person $x$ knows programming language $y$. |

| | | |
|---|---|---|
| domain | the domain is just your school | $\exists x(J(x) \land P(x) \land C(x))$ |
| one variable | the domain is all people | $\exists x(Y(x) \land J(x) \land P(x) \land C(x))$ |
| two variables | the domain is all people | $\exists x(Y(x) \land K(x, \text{Java}) \land K(x, \text{Prolog}) \land K(x, \text{C}++))$ |

d) Everyone in your class enjoys Thai food.

**Solution:**

| Predicate | Meaning |
|---|---|
| $Y(x)$ | $x$ is in your class. |
| $T(x)$ | $x$ enjoys Thai food. |
| $E(x, y)$ | person $x$ enjoys food from country $y$. |

| | | |
|---|---|---|
| domain | the domain is just your class | $\forall x(T(x))$ |
| one variable | the domain is all people | $\forall x(Y(x) \rightarrow T(x))$ |
| two variables | the domain is all people | $\forall x(Y(x) \rightarrow E(x, \text{Thai}))$ |

e) Someone in your class does not play Hockey.

**Solution:**

| Predicate | Meaning |
|---|---|
| $Y(x)$ | $x$ is in your school. |
| $H(x)$ | $x$ plays Hockey. |
| $P(x, y)$ | person $x$ plays sport $y$. |

| | | |
|---|---|---|
| domain | the domain is just your school | $\exists x(\neg H(x))$ |
| one variable | the domain is all people | $\exists x(Y(x) \wedge \neg H(x))$ |
| two variables | the domain is all people | $\exists x(Y(x) \wedge \neg P(x, \text{Hockey}))$ |