DEFINITION 1

A compound proposition that is always **true**, no matter what the truth values of the propositions that occur in it, is called a *tautology*.

A compound proposition that is always **false** is called a *contradiction*.

A compound proposition that is neither a **tautology** nor a **contradiction** is called a *contingency*.

TABLE 1.1 Example of a Tautology		
p	$\neg p$	$p \vee \neg p$
T	**F**	**T**
F	**T**	**T**
$p \vee \neg p$ **is a tautology because the compound proposition is true for every value of** p**.**		

TABLE 1.2 Example of a Contradiction		
p	$\neg p$	$p \wedge \neg p$
T	**F**	**F**
F	**T**	**F**

DEFINITION 2

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation, $p \equiv q$, denotes that p and q are logically equivalent.

TABLE 2 De Morgan's Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$

EXAMPLE 2

Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Solution: The truth table for these compound propositions is displayed in Table 3. Because the truth values of the compound propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ agree ***for all possible combinations*** of the truth values of p and q, it follows that $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$ is a tautology and these compound propositions are logically equivalent.

TABLE 3 Truth Tables for $\neg(p \wedge q)$ and $\neg p \vee \neg q$.							
p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$	$\neg p \vee \neg q$	$\neg p$	$\neg q$
T	**T**	**T**	**F**	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**T**	**T**	**F**	**T**
F	**T**	**F**	**T**	**T**	**T**	**T**	**F**
F	**F**	**F**	**T**	**T**	**T**	**T**	**T**
Tautology $\neg(p \wedge q) \equiv \neg p \vee \neg q$ **because** $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$ **is true for every value of** p **and** q**.**							

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \to q$					
p	q	$\neg p$	$\neg p \vee q$	$(\neg p \vee q) \leftrightarrow (p \to q)$	$p \to q$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	T	T
Tautology $\neg p \vee q \equiv p \to q$ **because** $(\neg p \vee q) \leftrightarrow (p \to q)$ **is true for every value of** p **and** q.					

EXAMPLE 3 Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution: The truth table for these compound propositions is displayed in Table 4.

p	$\neg p$	q	$p \to q$	$(p \to q) \leftrightarrow (\neg p \vee q)$	$\neg p \vee q$
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	T	T
$(p \to q) \equiv (\neg p \vee q)$ **because** $p \to q \leftrightarrow \neg p \vee q$ **is true for every value of** p **and** q.					

EXAMPLE 4 Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Solution: The truth table for these compound propositions is displayed in Table 5.

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent							
p	q	r	$(q \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q)$	$(p \vee r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ **because** $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$ **is true for every value of** p, q **and** r.							

TABLE 6. Logical Equivalences	
Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
TABLE 6. Logical Equivalences (continued)	
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Extended De Morgan's laws:

$$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n)$$
$$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$$

TABLE 7 Logical Equivalences Involving Conditional Statements
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditionals
$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$
$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg p$

EXAMPLE 5.1 Use De Morgan's laws to express the negation of "Miguel has a cellphone and he has a laptop computer."

Solution: Define propositions as shown below.

Variable	**Proposition**
p	"Miguel has a cellphone."
q	"Miguel has a laptop computer."
Original	**Compound Proposition**
$p \land q$	"Miguel has a cellphone ***and*** Miguel has a laptop computer."
Negation	
$\neg(p \land q)$	
Equivalent	
$\neg p \lor \neg q$	Miguel does ***not*** have a cellphone ***or*** he does ***not*** have a laptop computer.

EXAMPLE 5.2 Use De Morgan's laws to express the negation of "Heather will go to the concert or Steve will go to the concert."

Solution: Define propositions as shown below.

Variable	**Proposition**
r	"Heather will go to the concert."
s	"Steve will go the concert."
Original	**Compound Proposition**
$r \lor s$	"Heather will go to the concert ***or*** Steve will go to the concert."
Negation	
$\neg(r \lor s)$	
Equivalent	
$\neg r \land \neg s$	Heather will ***not*** go to the concert ***and*** Steve will ***not*** go to the concert.

EXAMPLE 6.1 Show that $\neg(p \rightarrow q)$ and $p \land \neg q$ are logically equivalent by developing a sequence of logical equivalencies.

Solution:

Expression	**Justification**
$\neg(p \rightarrow q)$	Original left-hand side
$\neg(\neg p \lor q)$	Example 3 and Table 4
$\neg\neg p \land \neg q$	De Morgan's laws
$p \land \neg q$	Double negation law

EXAMPLE 6.2

Show that $\neg(p \to q)$ and $p \land \neg q$ are logically equivalent by employing truth tables. Two compound propositions are equivalent if their equivalency is a tautology.

Solution:

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \leftrightarrow p \land \neg q$	$p \land \neg q$	$\neg q$
T	**T**	**T**	**F**	**T**	**F**	**F**
T	**F**	**F**	**T**	**T**	**T**	**T**
F	**T**	**T**	**F**	**T**	**F**	**F**
F	**F**	**T**	**F**	**T**	**F**	**T**

$\neg(p \to q)$ is logically equivalent to $p \land \neg q$ because $\neg(p \to q) \leftrightarrow p \land \neg q$ is true for every value of p and q.

EXAMPLE 7 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a sequence of logical equivalencies.

Solution:

Expression	Justification
$\neg(p \vee (\neg p \wedge q))$	Original left-hand side
$\neg p \wedge \neg(\neg p \wedge q)$	De Morgan's law $\neg(\boldsymbol{p} \vee \boldsymbol{q}) \equiv \neg\boldsymbol{p} \wedge \neg\boldsymbol{q}$
$\neg p \wedge (\neg\neg p \vee \neg q)$	De Morgan's laws $\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \equiv \neg\boldsymbol{p} \vee \neg\boldsymbol{q}$
$\neg p \wedge (p \vee \neg q)$	Double negation law $\neg(\neg\boldsymbol{p}) \equiv \boldsymbol{p}$
$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distributive law $\boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r}) \equiv (\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\boldsymbol{p} \wedge \boldsymbol{r})$
$(\mathbf{F}) \vee (\neg p \wedge \neg q)$	Negation law $p \wedge \neg p \equiv \mathbf{F}$
$(\neg p \wedge \neg q) \vee \mathbf{F}$	Commutative law $p \vee q \equiv q \vee p$
$\neg p \wedge \neg q$	Identity law $p \vee \mathbf{F} \equiv p$

EXAMPLE 8 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to $\mathbf{T}$. (*Note*: This could also be done using a truth table.)

Expression	Justification
$(p \wedge q) \rightarrow (p \vee q)$	Original expression
$\neg(p \wedge q) \vee (p \vee q)$	Table 3 $(\boldsymbol{p} \rightarrow \boldsymbol{q}) \equiv (\neg\boldsymbol{p} \vee \boldsymbol{q})$
$(\neg p \vee \neg q) \vee (p \vee q)$	De Morgan's law $\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \equiv \neg\boldsymbol{p} \vee \neg\boldsymbol{q}$
$\neg p \vee \neg q \vee p \vee q$	Restatement removing parenthesis
$\neg p \vee p \vee \neg q \vee q$	Commutative law $p \vee q \equiv q \vee p$
$(\neg p \vee p) \vee (\neg q \vee q)$	Associative law $(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \vee \neg p) \vee (q \vee \neg q)$	Commutative law $p \vee q \equiv q \vee p$ applied twice
$\mathbf{T} \vee \mathbf{T}$	Negation law $p \vee \neg p \equiv \mathbf{T}$ applied twice.
$\mathbf{T}$	Domination law $p \vee \mathbf{T} \equiv \mathbf{T}$

4. Use a truth table to verify the associative law
   a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

   Solution:

   1. **How many rows are needed? Answer: $2^{|P|} + 1 = 8 + 1 = 9$ rows, where $|P| =$ the number of propositions.**
   2. **How many columns do we need? Let us develop the truth table a step at a time. First we need a column for each of the propositional variables, p, q, and r.**

p	q	r

   3. **Now we need to populate the rows for the columns labeled p, q, and r. Alternate Ts and Fs in the r-column.**

p	q	r
		T
		F
		T
		F
		T
		F
		T
		F

   4. **Proceed to alternate Ts and Fs for the q- and p-columns but halve the frequency as you move to the left.**

p	q	r
	T	T
	T	F
	F	T
	F	F
	T	T
	T	F
	F	T
	F	F

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

5. Now make a column for the first operation, $(p \vee q)$.

p	q	r	$(p \vee q)$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

6. Compute $(p \vee q)$ in every row.

p	q	r	$(p \vee q)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

7. Now make a column for the second operation, $(p \vee q) \vee r$

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$
T	T	T	T	
T	T	F	T	
T	F	T	T	
T	F	F	T	
F	T	T	T	
F	T	F	T	
F	F	T	F	
F	F	F	F	

8. As before, compute the value of $(p \vee q) \vee r$ for every row.

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

8. Now that we have computed all the truth-values for the left side of the equivalence, we must consider what to do next. Two expressions are equivalent if their biconditional values are true for every value of the propositional variables. Now, make a column having the original expression but substituting the biconditional operator for the equivalence operator.

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
T	T	T	T	T	
T	T	F	T	T	
T	F	T	T	T	
T	F	F	T	T	
F	T	T	T	T	
F	T	F	T	T	
F	F	T	F	T	
F	F	F	F	F	

9. Make two more columns. The rightmost column is labeled $(q \vee r)$ and the column between the rightmost column and the biconditional column is labeled $p \vee (q \vee r)$.

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$p \vee (q \vee r)$	$(q \vee r)$
T	T	T	T	T			
T	T	F	T	T			
T	F	T	T	T			
T	F	F	T	T			
F	T	T	T	T			
F	T	F	T	T			
F	F	T	F	T			
F	F	F	F	F			

10. Complete the rightmost column $(q \vee r)$.

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$p \vee (q \vee r)$	$(q \vee r)$
T	T	T	T	T			T
T	T	F	T	T			T
T	F	T	T	T			T
T	F	F	T	T			F
F	T	T	T	T			T
F	T	F	T	T			T
F	F	T	F	T			T
F	F	F	F	F			F

11. Complete the column $p \vee (q \vee r)$.

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$p \vee (q \vee r)$	$(q \vee r)$
T	T	T	T	T		T	T
T	T	F	T	T		T	T
T	F	T	T	T		T	T
T	F	F	T	T		T	F
F	T	T	T	T		T	T
F	T	F	T	T		T	T
F	F	T	F	T		T	T
F	F	F	F	F		F	F

12. Complete the biconditional column $(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$.

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$p \vee (q \vee r)$	$(q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	T	F	F

14. If every value in the biconditional column is True, then the two compound propositions are equivalent. Add an additional row forming the conclusion that we just stated.

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$	$p \vee (q \vee r)$	$(q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	T	F	F
$(p \vee q) \vee r$ is equivalent to $p \vee (q \vee r)$ because both compound propositions yield a value of true for every combination of p, q, and r.							

15. For the purpose of submitting an assignment, we need only submit the finished table as shown in step 14 above.