Good morning. This lecture addresses applications of propositional logic. We discuss several examples and conclude with several exercises from our text that are similar to those that are assigned.

Translating English Sentences

EXAMPLE 1 How can this English sentence be translated into a logical expression?

> "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

Solution: Define the following propositions:

Proposition	**English statement**
a	"You can access the Internet from campus."
c	"You are a computer science major."
f	"You are a freshman"

"only if" maps to $\rightarrow$

$$a \rightarrow (c \vee \neg f)$$

EXAMPLE 2 Translate the following sentence into a logical expression.

> "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Solution: Define the following propositions:

Proposition	**English statement**
q	"You can ride the roller coaster."
r	"You are under 4 feet tall."
s	"You are older than 16 years old."

$$(r \wedge \neg s) \rightarrow \neg q$$

System Specification

Translating sentences in natural language (such as English) into logical expressions is an essential part of specifying both hardware and software systems.

EXAMPLE 3 Express the specification "The automated reply cannot be sent when the file system is full" using logical connectives.

Solution: Define the following propositions:

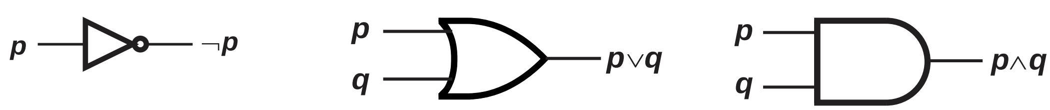
Proposition	English statement
p	"The automated reply can be sent."
$\neg p$	"It is not the case that the automated reply can be sent."
$\neg p$	"The automated reply cannot be sent."
q	"The file system in full."

$$q \rightarrow \neg p$$

Logic Circuits

Propositional logic can be applied to the design of computer hardware.

A logic circuit (or digital circuit) receives input signals $p_1, p_2, \cdots, p_n$, each a bit [either 0 (off) or 1 (on)], and produces output signals $s_1, s_2, \cdots, s_m$, each a bit.



Inverter | **OR** gate | **AND** gate

FIGURE 1 Basic logic gates

Complicated digital circuits can be constructed from three basic circuits, called gates, shown in Figure 1. The inverter, or **NOT gate**, takes an input bit p, and produces an output $\neg p$. The **OR gate** takes two input signals p and q, each a bit, and produces an as output the signal $p \vee q$. Finally, the **AND gate** takes two input signals p and q, each a bit, and produces the output signal $p \wedge q$.

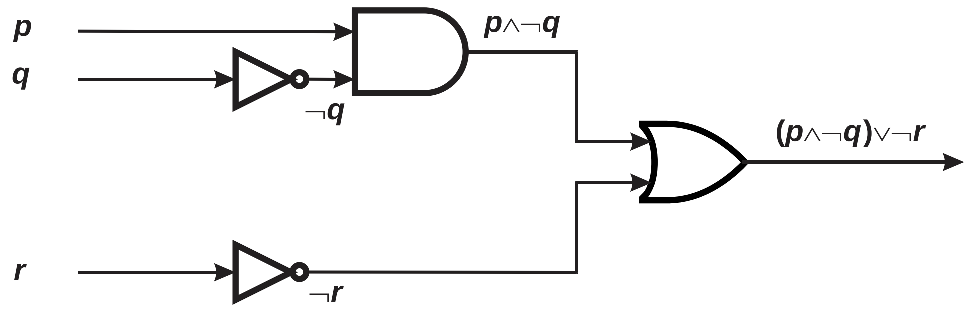
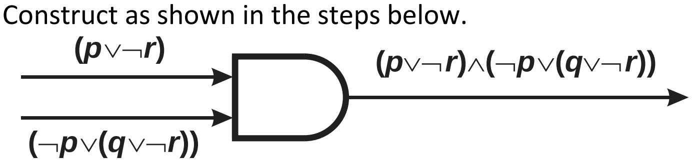


FIGURE 2 A combinatorial circuit

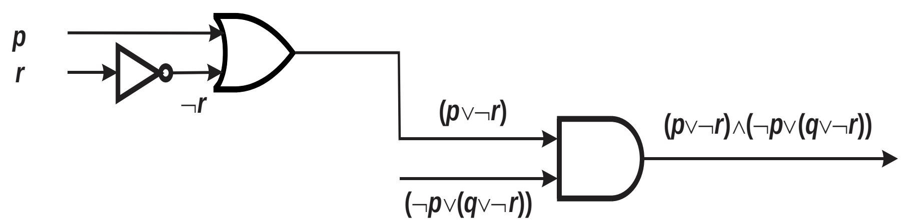
EXAMPLE 9 Determine the output for the combinatorial circuit in Figure 2.
Solution: In Figure 2 we display the output of each logic gate in the circuit. We see that the AND gate takes input of p and $\neg q$, the output of the inverter with input q, and produces $p \wedge \neg q$. Next, we note that the OR gate takes input $p \wedge \neg q$ and $\neg r$, the output of the inverter with input r, and produces the final output $(p \wedge \neg q) \vee \neg r$.

EXAMPLE 10 Build a digital circuit that produces the output $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$
Solution: Construct as shown in the steps below.

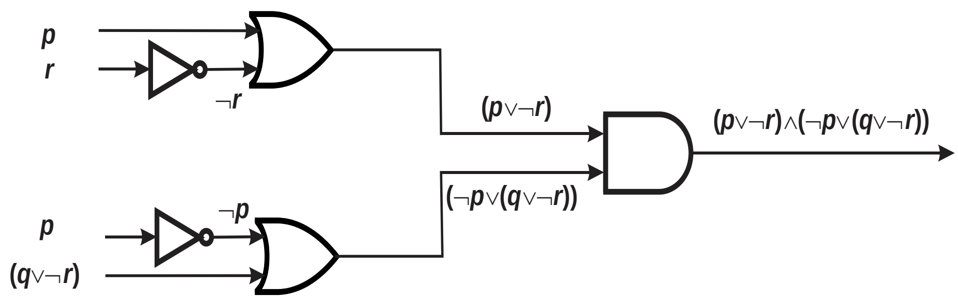
Step 1



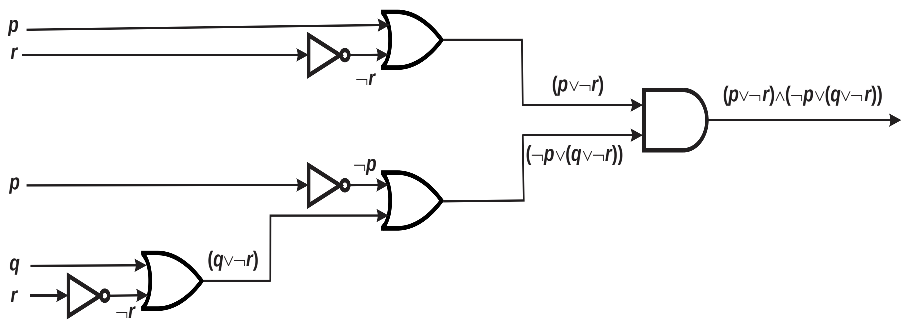
Step 2



Step 3



Step 4



Logic and Bit Operations

Truth Value	*Bit*
T	1
F	0

REMARK A *bit* is the contraction of *binary digit*, a zero (0) or a one (1).

DEFINITION 7 A *bit string* is a sequence of zero or more bits. The *length* of the string is the number of bits in the string.

EXAMPLE 12 1010 1001 1 is a bit string of length nine.

EXAMPLE 13 Find the bitwise *OR*, bitwise *AND*, and bitwise *XOR*, of the bit strings 01 1011 0110 and 11 0001 1101.

Solution:

	0	1	1	0	1	1	0	1	1	0
	1	1	0	0	0	1	1	1	0	1
OR	1	1	1	0	1	1	1	1	1	1
	0	1	1	0	1	1	0	1	1	0
	1	1	0	0	0	1	1	1	0	1
AND	0	1	0	0	0	1	0	1	0	0
	0	1	1	0	1	1	0	1	1	0
	1	1	0	0	0	1	1	1	0	1
XOR	1	0	1	0	1	0	1	0	1	1

In Exercises 1 – 6, translate the given statement into propositional logic using the propositions provided.

3. You can graduate only if you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book. Express your answer in terms of g: "You can graduate," m: "You owe money to the university," r: "You have completed the requirements of your major," and b: "You have an overdue library book."

Solution:

Propositional Variable	Proposition
g	You can graduate.
m	You owe money to the university.
r	You have completed the requirements of your major.
b	You have an overdue library book.

1. **Dissect the statement:**

   You can graduate only if you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book.

2. Consider first the phrase "You can graduate only if." Recall, from the previous lecture that "p only if q" is one of the alternatives for $p \to q$. In this case, "You can graduate" is the *premise, p,* and "you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book" is the conclusion, q. Thus, we form the partial conditional statement:

$$g \to$$

3. Next, we consider the phrase:

   … you have completed the requirements of your major *and* you do not owe money to the university *and* you do not have an overdue library book.

   In particular, we note that we have the conjunction of three separate propositions and we form the partial conditional statement:

$$g \to (\ \wedge\ \wedge\ )$$

4. Now, we consider the phrase "you have completed the requirements of your major" and note that it is exactly our proposition r. We revise our partial conditional statement to:

$$g \to (r\ \wedge\ \wedge\ )$$

5. We consider the next phrase "you do not owe money to the university" and note that it is the negation of our proposition m. We revise our partial conditional statement to:

$$g \rightarrow (r \land \neg m \land \ )$$

6. Finally, we consider the phrase "you do not have an overdue library book" and note that it is the negation of our proposition b. We complete our conditional statement to:

$$g \rightarrow (r \land \neg m \land \neg b)$$

7. We present our work in a manner that can be easily understood.

Propositional Variable	Proposition
g	You can graduate.
m	You owe money to the university.
r	You have completed the requirements of your major.
b	You have an overdue library book.

Compound Proposition	English Equivalent
$g \ \rightarrow$	You can graduate only if
$g \ \rightarrow (\ \land\ \land\ )$	You can graduate only if … *and* … *and* … .
$g \ \rightarrow (r \land\ \land\ )$	You can graduate only if you have completed the requirements of your major *and* … *and* … .
$g \ \rightarrow (r \land \neg m \land\ )$	You can graduate only if you have completed the requirements of your major *and* you do not owe money to the university *and* … .
$g \ \rightarrow (r \land \neg m \land \neg b)$	You can graduate only if you have completed the requirements of your major *and* you do not owe money to the university *and* you do not have an overdue library book.

7. Express these system specifications using the propositions p "The message is scanned for viruses" and q "The message was sent from an unknown system." and logical connectives (including negations).

Solution:

Propositional Variable	Proposition
p	The message is scanned for viruses.
q	The message was sent from an unknown system.

Part	Specification
b	"The message was sent from an unknown system but it was not scanned for viruses."

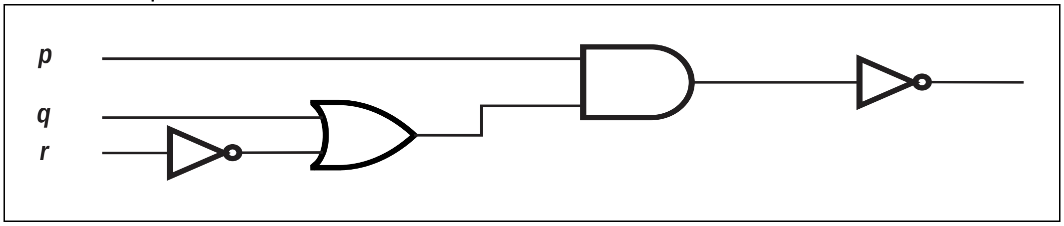
1. We identify the phrase "The message was sent from an unknown system" as q.
2. We identify the phrase "it was not scanned for viruses" as equivalent to the negation of "The message is scanned for viruses." The phrase, then, is equivalent to: $\neg p$.
3. We examine the connective "but" and determine that in this case "but" means "and."
4. We construct the compound proposition:

$$q \wedge \neg p$$

5. We present our work so that it can be easily understood.

Part	Specification	Expression
b	"The message was sent from an unknown system but it was not scanned for viruses."	$q \wedge \neg p$

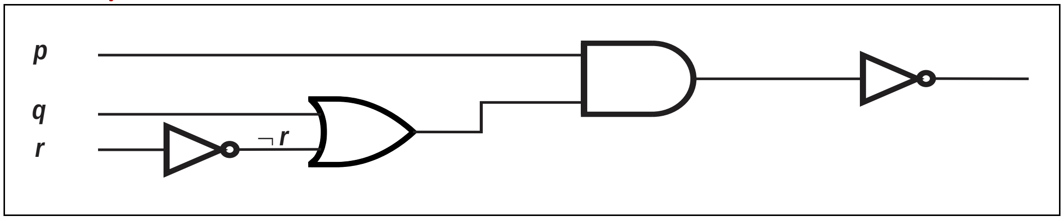
41. Find the output of this combinatorial circuit.



Solution:

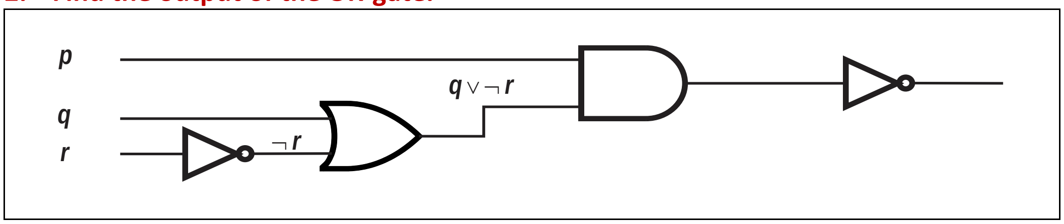
Label input lines, moving from left to right.

1. **Complement r.**



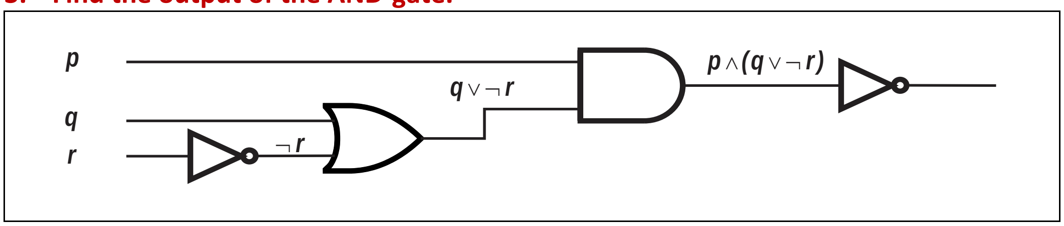
$\neg r$

2. **Find the output of the OR-gate.**



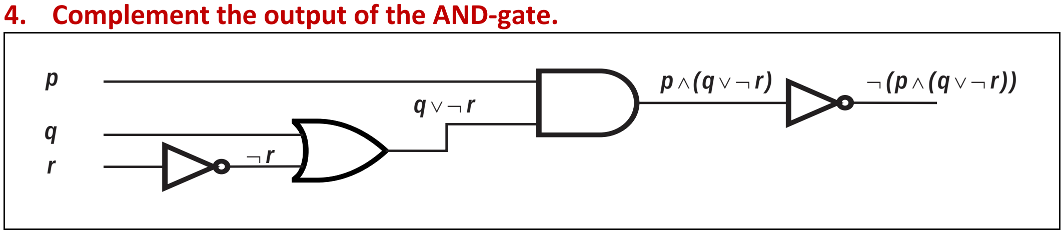
$q \vee \neg r$

3. **Find the output of the AND-gate.**



$p \wedge (q \vee \neg r)$

4. **Complement the output of the AND-gate.**



$\neg(p \wedge (q \vee \neg r))$