Mathematical Induction

- Mathematical induction can be used to prove statements that assert \( P(n) \) is true for all positive integers \( n \), where \( P(n) \) is a propositional function. A proof by mathematical induction has two parts
  - a basis step, where we show \( P(1) \) is true and
  - an inductive step, where we show that for all positive integers \( k \), if \( P(k) \) is true, then \( P(k + 1) \) is true.

Mathematical Induction

**PRINCIPLE OF MATHEMATICAL INDUCTION.** To prove \( P(n) \) is true for all positive integers \( n \), where \( P(n) \) is a propositional function, we complete two steps.

1. **BASIS STEP.** We verify that \( P(1) \) is true.
2. **INDUCTIVE STEP.** We show that the conditional statement \( P(k) \rightarrow P(k + 1) \) is true for all positive integers \( k \).

Remark

Expressed as a rule of inference, this proof technique can be stated as

\[
\left[P(1) \land \forall k \left( P(k) \rightarrow P(k + 1) \right) \right] \rightarrow \forall n P(n)
\]

Remark

In a proof by mathematical induction it is not assumed that \( P(k) \) is true for all positive integers! It is only shown that if it is assumed that \( P(k) \) is true, then \( P(k + 1) \) is also true. Thus, a proof by mathematical induction is not a case of begging the question, or circular reasoning.

**EXAMPLE 1**

Show that if \( n \) is a positive integer, then

\[
1 + 2 + \cdots + n = \frac{n(n + 1)}{2}
\]

**Solution:**

**BASIS STEP:** \( P(1) \) is true, because 

\[
1 = \frac{1(1+1)}{2}
\]

**INDUCTIVE STEP:** For the inductive hypothesis we assume that \( P(k) \) holds for an arbitrary positive integer \( k \). That is, we assume that

\[
1 + 2 + \cdots + k = \frac{k(k + 1)}{2}
\]

Now, we must prove that

\[
1 + 2 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 2)}{2}
\]

1. Apply the inductive hypothesis to the left side

\[
\frac{k(k + 1)}{2} + (k + 1)
\]

2. Algebraically transform our result until it matches the right hand side

\[
\frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}
\]
EXAMPLE 2  Find a formula for the sum of the positive odd integers. Then prove that your formula is correct using mathematical induction.

Solution: After fussin’ some, we find \[ \sum_{i=1}^{n} (2i - 1) = n^2. \] Please refer to fussing below.

<table>
<thead>
<tr>
<th>Integer</th>
<th>Sum</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1+3</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1+3+5</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>1+3+5+7</td>
<td>16</td>
</tr>
<tr>
<td>( n )</td>
<td>( \sum_{i=1}^{n} (2i - 1) )</td>
<td>( n^2 )</td>
</tr>
</tbody>
</table>

BASIS STEP: \( P(1) \) is true, because \( 1 = 2(1) - 1 = 1 \)

INDUCTIVE STEP: For the inductive hypothesis we assume that \( P(k) \) holds for an arbitrary positive integer \( k \). That is, we assume that
\[
1 + 3 + \cdots + 2k - 1 = k^2
\]
Now, we must prove that
\[
1 + 3 + \cdots + 2k - 1 + 2(k + 1) - 1 = (k + 1)^2
\]
1. Apply the inductive hypothesis to the left side
\[
k^2 + 2(k + 1) - 1
\]
2. Algebraically transform our result until it matches the right hand side
\[
k^2 + 2(k + 1) - 1 = k^2 + 2k + 2 - 1
\]
\[
= k^2 + 2k + 1
\]
\[
= (k + 1)^2
\]
EXAMPLE 3  Use mathematical induction to show that
\[ 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \]

Solution: Let \( P(n) \) be the proposition that \( 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \) for the integer \( n \).

**BASIS STEP:** \( P(0) \) is true because \( 2^0 = 2^1 - 1 = 1 \)

**INDUCTIVE STEP:** For the inductive hypothesis we assume that \( P(k) \) holds for an arbitrary positive integer \( k \). That is, we assume that
\[ 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 \]
Now, we must prove that
\[ 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1 \]

1. Apply the inductive hypothesis to the left side
   \[ 2^{k+1} - 1 + 2^{k+1} = \]

2. Algebraically transform our result until it matches the right hand side
   \[ 2 \cdot 2^{k+1} - 1 = \]
   \[ 2^{k+1} - 1 = \]
   \[ 2^{k+2} - 1 = 2^{k+2} - 1 \]