DEFINITION 1  Let $A$ and $B$ be nonempty sets. A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$. We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element $a$ of $A$. If $f$ is a function from $A$ to $B$, we write $f : A \rightarrow B$.

EXAMPLE 0.1  Let $A = \{a, b, c, d, e\}$, $B = \{0, 1, 2, 3, 4\}$, and $f : A \rightarrow B$ is defined by the relation $R = \{(a, 0), (b, 1), (c, 2), (d, 3)\}$ where $(u, v) \rightarrow f(u) = v$ in $R$. Thus $f(a) = 0, f(b) = 1, f(c) = 2, \text{and } f(d) = 3$. Is $f : A \rightarrow B$ a function?

\[ a \quad \rightarrow \quad 0 \quad \quad \quad b \quad \rightarrow \quad 1 \quad \quad \quad c \quad \rightarrow \quad 2 \quad \quad \quad d \quad \rightarrow \quad 3 \quad \quad \quad e \quad \rightarrow \quad 4 \]

Solution: No. Function $f$ fails to assign element $e \in A$ to any element in $B$. Recall, the definition “A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A.”$ Since $e \in A$, an element of $B$ must be assigned to $e$ and no such assignment is defined by $f : A \rightarrow B$.  


EXAMPLE 0.2  Let $A = \{a, b, c, d, e\}, B = \{0, 1, 2, 3, 4\}$, and $f: A \rightarrow B$ is defined by the relation $R = \{(a, 0), (a, 1), (b, 2), (c, 3), (d, 4), (e, 0)\}$ where $(u, v) \rightarrow f(u) = v$ in $R$.

Thus $f(a) = 0, f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4, \text{ and } f(e) = 0$. Is $f: A \rightarrow B$ a function?

**Solution:** No. Function $f$ fails to assign exactly one element of $B$ to element $a \in A$. Recall, the definition “A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A$.” Since $f(a) = 0$ and $f(a) = 1$, $f: A \rightarrow B$ is not a function.

**Remark:** Functions are sometimes also called mappings or transformations.

**DEFINITION 2** If $f$ is a function from $A$ to $B$, we say that $A$ is the domain of $f$ and $B$ is the codomain of $f$. If $f(a) = b$, we say that $b$ is the image of $a$ and $a$ is a preimage of $b$. The range of $f$ is the set of all images of elements of $A$. Also, if $f$ is a function from $A$ to $B$, we say that $f$ maps $A$ to $B$. 

Figure 1 Assignment of Grades in a Discrete Mathematics Class
EXAMPLE 1  What are the domain, codomain, and range of the function that assigns grades to students shown in Figure 1.

Solution:  Let \( G \) be the function that assigns a grade to a student in our discrete mathematics class.  Note that \( G(Adams) = A \), for instance.

- The domain of \( G \) is the set \{Adams, Chou, Goodfriend, Rodriguez, Stevens\}.
- The codomain is the set \{A,B,C,D,F\}.
- The range is the set \{A, B, C, F\} because each grade except \( D \) is assigned to some student.

EXAMPLE 2  Let \( R \) be the relation consisting of ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22), where each pair consists of a graduate student and the age of this student.  What is the function that this relation determines?

Solution:  The relation defines the function \( f \) as given below.

- \( f(Abdul) = 22 \)
- \( f(Brenda) = 24 \)
- \( f(Carla) = 21 \)
- \( f(Desire) = 22 \)
- \( f(Eddie) = 24 \)
- \( f(Felicia) = 22 \)
- The domain of \( f \) is the set \{Abdul, Brenda, Carla, Desire, Eddie, Felicia\}
- A codomain of \( f \) is the set of positive integers.
- The range of \( f \) is the set \{21,22,24\}

EXAMPLE 4  Let \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) assign the square of an integer to this integer.  Further define \( f \) and identify the domain, codomain, and the range of \( f \).

Solution

- \( f(x) = x^2 \)
- The domain of \( f \) is \( \mathbb{Z} \), the set of all integers.
- The codomain of \( f \) is \( \mathbb{Z} \), the set of all integers.
- The range of \( f \) is the set of all integers that are perfect squares namely, \{0, 1, 4, 9, \ldots \}

DEFINITION 3  Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then \( f_1 + f_2 \) and \( f_1 f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x) \\
(f_1 f_2)(x) = f_1(x) f_2(x)
\]

EXAMPLE 6  Let \( f_1 \) and \( f_2 \) be functions from \( \mathbb{R} \) to \( \mathbb{R} \) such that \( f_1(x) = x^2 \) and \( f_2(x) = x - x^2 \).  What are the functions \( f_1 + f_2 \) and \( f_1 f_2 \)?

Solution

- \( (f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x \)
- \( (f_1 f_2)(x) = f_1(x) f_2(x) = x^2(x - x^2) = x^3 - x^4 \)
DEFINITION 4 Let \( f \) be a function from the set \( A \) to the set \( B \) and let \( S \) be a subset of \( A \). The image of \( S \) under the function \( f \) is the subset of \( B \) that consists of the images of \( S \). We denote the image of \( S \) by \( f(S) \), so
\[
\text{for } S \subseteq A, f(S) = \{ t \mid \exists s \in S \text{ such that } t = f(s) \}.
\]
We also use the shorthand \( \{ f(s) \mid s \in S \} \) to denote this set.

 Remark: The notation \( f(S) \) for the image of the set \( S \) under the function \( f \) is potentially ambiguous. Here, \( f(S) \) denotes a set, and not the value of the function \( f \) for the set \( S \).

EXAMPLE 7 Let \( A = \{ a, b, c, d, e \} \) and \( B = \{1,2,3,4\} \) with \( f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1, \) and \( f(e) = 1 \). What is the image of the subset \( S = \{ b, c, d \} \)?
Solution: The image of \( S \) under the function \( f \) is the set \( f(S) = \{1,4\} \).

One-to-one and Onto Functions

DEFINITION 5 A function \( f \) is said to be one-to-one, or injective, if and only if \( f(a) = f(b) \) implies that \( a = b \) for all \( a \) and \( b \) in the domain of \( f \). A function is said to be an injection if it is one-to-one.

EXAMPLE 8 Determine whether the function \( f \) from \( \{ a, b, c, d \} \) to \( \{1,2,3,4,5\} \) with \( f(a) = 4, f(b) = 5, f(c) = 1, \) and \( f(d) = 3 \) is one-to-one.
Solution: The function \( f \) is one-to-one because \( f \) takes on different values at the four elements of its domain. This is illustrated in Figure 3.

![Figure 3 A One-to-One Function](image)

EXAMPLE 9 Determine whether the function \( f(x) = x^2 \) from the set of integers to the set of integers is one-to-one.
Solution: The function \( f(x) = x^2 \) is not one-to-one because, for instance, \( f(1) = f(-1) = 1 \), but \( 1 \neq -1 \).

EXAMPLE 10 Determine whether the function \( f(x) = x + 1 \) from the set of real numbers to itself is one-to-one.
Solution: The function \( f(x) = x + 1 \) is a one-to-one function. To demonstrate this, note that \( x + 1 \neq y + 1 \) when \( x \neq y \).
DEFINITION 6 A function $f$ whose domain and codomain are subsets of the set of real numbers is called \textit{increasing} if $f(x) \leq f(y)$, and \textit{strictly increasing} if $f(x) < f(y)$, whenever $x < y$ and $x$ and $y$ are in the domain of $f$. Similarly, $f$ is called \textit{decreasing} if $f(x) \geq f(y)$, and \textit{strictly decreasing} if $f(x) > f(y)$, whenever $x < y$ and $x$ and $y$ are in the domain of $f$. (The word \textit{strictly} in this definition indicates a strict inequality.)

\textbf{Remark:} A function that is either strictly increasing or strictly decreasing is guaranteed to be one-to-one.

DEFINITION 7 A function $f$ from $A$ to $B$ is called \textit{onto}, or \textit{surjective}, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function $f$ is called a \textit{surjection} if it is onto.

![Graphs of functions](image)

Figure 5 (a) One-to-one, not onto
Figure 5 (b) Onto, not one-to-one
Figure 5 (c) One-to-one and onto
Figure 5 (d) Neither one-to-one nor onto
Figure 5 (e) Not a function
EXAMPLE 11  Let $f$ be the function from $\{a, b, c, d\}$ to $\{1,2,3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is $f$ an onto function? 
Solution: Because all three elements of the codomain are images of elements in the domain, we see that $f$ is onto. This is illustrated in the figure below. Note that if the codomain were $\{1,2,3,4\}$, then $f$ would not be onto.

![Diagram of function](image)

EXAMPLE 12  Is the function $f(x) = x^2$ from the set of integers to the set of integers onto? 
Solution: The function $f$ is not onto because there is no integer $x$ with $x^2 = -1$.

EXAMPLE 13  Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto? 
Solution: The function is onto, because for every integer $y$ there is an integer $x$ such that $f(x) = y$. To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$.

DEFINITION 8  The function $f$ is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.

EXAMPLE 14  Let $f$ be the function from $\{a, b, c, d\}$ to $\{1,2,3,4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is $f$ a bijection? 
Solution: The function $f$ is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, $f$ is a bijection.

EXAMPLE 15  Let $A$ be a set, the identity function on $A$ is the function $i_A : A \to A$, where $i_A(x) = x$ for all $x \in A$. In other words, the identity function $i_A$ is the function that assigns each element to itself. The function $i_A$ is one-to-one and onto, so it is a bijection.

**Inverse Functions and Compositions of Functions**

DEFINITION 9  Let $f$ be a one-to-one correspondence from the set $A$ to the set $B$. The inverse function of $f$ is the function that assigns to an element $b$ belonging to $B$ the unique element $a$ in $A$ such that $f(a) = b$. The inverse function is denoted by $f^{-1}$. Hence, $f^{-1}(b) = a$ when $f(a) = b$. 
EXAMPLE 17 Let \( f: \mathbb{Z} \to \mathbb{Z} \) be such that \( f(x) = x + 1 \). Is \( f \) invertible, and if it is, what is the inverse?

Solution: The function \( f \) has an inverse because it is a one-to-one correspondence, as has been shown previously. To reverse the correspondence, suppose that \( y \) is the image of \( x \), so that \( y = x + 1 \). Then \( x = y - 1 \). This means that \( y - 1 \) is the unique element of \( \mathbb{Z} \) that is sent to \( y \) by \( f \). Consequently, \( f^{-1}(y) = y - 1 \).

EXAMPLE 18 Let \( f \) be the function from \( \mathbb{R} \) to \( \mathbb{R} \) with \( f(x) = x^2 \). Is \( f \) invertible?

Solution: Because \( f(2) = f(-2) = 4 \), \( f \) is not one-to-one. If an inverse function were defined it would have to assign two elements to 4. Hence, \( f \) is not invertible.

DEFINITION 10 Let \( g \) be a function from the set \( A \) to the set \( B \) and let \( f \) be a function from the set \( B \) to the set \( C \). The composition of the functions \( f \) and \( g \), denoted by \( f \circ g \), is defined by

\[ (f \circ g)(a) = f(g(a)) \]

EXAMPLE 20 Let \( g \) be the function from the set \( \{a, b, c\} \) to itself such that \( g(a) = b, g(b) = c, \) and \( g(c) = a \). Let \( f \) be the function from the set \( \{a, b, c\} \) to the set \( \{1, 2, 3\} \) such that \( f(a) = 3, f(b) = 2, \) and \( f(c) = 1 \). What is the composition of \( f \) and \( g \), and what is the composition of \( g \) and \( f \)?

Solution: The composition \( f \circ g \) is defined by \( (f \circ g)(a) = f(g(a)) = f(2) = 2, (f \circ g)(b) = f(g(b)) = f(1) = 1, \) and \( (f \circ g)(c) = f(g(c)) = f(3) = 3 \)

EXAMPLE 21 Let \( f \) and \( g \) be the functions from the set of integers to the set of integers defined by \( f(x) = 2x + 3 \) and \( g(x) = 3x + 2 \). What is the composition of \( f \) and \( g \)? What is the composition of \( g \) and \( f \)?

Solution: Both the compositions \( f \circ g \) and \( g \circ f \) are defined. Moreover,

\[ (f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7 \]

and

\[ (g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11 \]

Remark: Note that even though \( f \circ g \) and \( g \circ f \) are defined for functions \( f \) and \( g \) in Example 21, \( f \circ g \) and \( g \circ f \) are not equal. In other words, the commutative law does not hold for the composition of functions.

Inverse Functions: When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that \( f \) is a one-to-one correspondence from the set \( A \) to the set \( B \). Then the inverse function \( f^{-1} \) exists and is a one-to-one correspondence from \( B \) to \( A \). The inverse function reverses the correspondence of the original function, so \( f^{-1}(b) = a \) when \( f(a) = b \), and \( f(a) = b \) when \( f^{-1}(b) = a \). Hence,

\[ (f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a, \]

and
\[(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.\]

The Graphs of Functions

Some Important Functions