Discrete Structures   Lecture 11
CMSC 2123       2.3 Functions

Introduction
Good morning. In this section we study functions. A function is a mapping from one set to another
set or, perhaps, from one set to itself. We study the properties of functions. A mapping may not
be a function. A function may or may not be invertible. We want certain functions to be invertible.
For example, we want a function that encrypts a message to be invertible so that we can decrypt
the message. On the other hand, we want our passwords to be impossible to decrypt – that is –
we want the function that maps our password into a binary code to be not invertible.

DEFINITION 1    Let A and B be nonempty sets. A function f from A to B is an assignment of
exactly one element of B to each element of A. We write \( f(a) = b \) if b is the
unique element of B assigned by the function f to the element a of A. If f is
a function from A to B, we write \( f: A \rightarrow B \).

EXAMPLE 0.1    Let \( A = \{a, b, c, d, e\} \), \( B = \{0,1,2,3,4\} \), and \( f: A \rightarrow B \) is defined by the
relation \( R = \{(a, 0), (b, 1), (c, 2), (d, 3)\} \) where \( (u, v) \rightarrow f(u) = v \) in R.
Thus \( f(a) = 0, f(b) = 1, f(c) = 2, \) and \( f(d) = 3 \).

Is \( f: A \rightarrow B \) a function?

\[
\begin{array}{c}
\text{a} & \rightarrow & \bullet 0 \\
\text{b} & \rightarrow & \bullet 1 \\
\text{c} & \rightarrow & \bullet 2 \\
\text{d} & \rightarrow & \bullet 3 \\
\text{e} & \rightarrow & \bullet 4
\end{array}
\]

Solution:  \textbf{Nope.} Function f fails to assign element \( e \in A \) to any element in B.
Recall, the definition “A function f from A to B is an assignment of exactly one
element of B to each element of A.” Since \( e \in A \), an element of B must be
assigned to e and no such assignment is defined by \( f: A \rightarrow B \).
EXAMPLE 0.2 Let $A = \{a, b, c, d, e\}$, $B = \{0, 1, 2, 3, 4\}$, and $f: A \rightarrow B$ is defined by the relation $R = \{(a, 0), (a, 1), (b, 2), (c, 3), (d, 4), (e, 0)\}$ where $(u, v) \rightarrow f(u) = v$ in $R$. Thus $f(a) = 0, f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4,$ and $f(e) = 0$.

Is $f: A \rightarrow B$ a function?

Solution: Nope. Function $f$ fails to assign exactly one element of $B$ to element $a \in A$. Recall, the definition “A function $f$ from $A$ to $B$ is an assignment of exactly one element of $B$ to each element of $A.”$ Since $f(a) = 0$ and $f(a) = 1, f: A \rightarrow B$ is not a function.

Remark: Functions are sometimes also called mappings or transformations.

DEFINITION 2 If $f$ is a function from $A$ to $B$, we say that $A$ is the domain of $f$ and $B$ is the codomain of $f$. If $f(a) = b$, we say that $b$ is the image of $a$ and $a$ is a preimage of $b$. The range of $f$ is the set of all images of elements of $A$. Also, if $f$ is a function from $A$ to $B$, we say that $f$ maps $A$ to $B$. 

FIGURE 1 Assignment of Grades in a Discrete Mathematics Class
EXAMPLE 1
What are the domain, codomain, and range of the function that assigns grades to students shown in Figure 1.

Solution: Let $G$ be the function that assigns a grade to a student in our discrete mathematics class. Note that $G(\text{Adams}) = A$, for instance.
- The domain of $G$ is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}.
- The codomain is the set {A, B, C, D, F}.
- The range is the set {A, B, C, F} because each grade except D is assigned to some student.

EXAMPLE 2
Let $R$ be the relation consisting of ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22), where each pair consists of a graduate student and the age of this student. What is the function that this relation determines?

Solution: The relation defines the function $f$ as given below.
- $f($Abdul$)=22$
- $f($Brenda$)=24$
- $f($Carla$)=21$
- $f($Desire$)=22$
- $f($Eddie$)=24$
- $f($Felicia$)=22$
- The domain of $f$ is the set {Abdul, Brenda, Carla, Desire, Eddie, Felicia}
- A codomain of $f$ is the set of positive integers.
- The range of $f$ is the set {21, 22, 24}
EXAMPLE 3 Let $f$ be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$. What are the domain, codomain, and range of $f$?

Solution:
- The domain of $f$ is the set of all strings of length 2 or greater.
- The codomain $f$ is the set $\{00, 01, 10, 11\}$.
- The range of $f$ is the same as the codomain of $f$, namely the set $\{00, 01, 10, 11\}$.

EXAMPLE 4 Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ assign the square of an integer to this integer. Further define $f$ and identify the domain, codomain, and the range of $f$.

Solution
- $f(x) = x^2$
- The domain of $f$ is $\mathbb{Z}$, the set of all integers.
- The codomain of $f$ is $\mathbb{Z}$, the set of all integers.
- The range of $f$ is the set of all integers that are perfect squares namely, $\{0, 1, 4, 9, \ldots \}$

DEFINITION 3 Let $f_1$ and $f_2$ be functions from $A$ to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from $A$ to $\mathbb{R}$ defined by

- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1 f_2)(x) = f_1(x)f_2(x)$

EXAMPLE 6 Let $f_1$ and $f_2$ be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution
- $(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$
- $(f_1 f_2)(x) = f_1(x)f_2(x) = x^2(x - x^2) = x^3 - x^4$

DEFINITION 4 Let $f$ be a function from the set $A$ to the set $B$ and let $S$ be a subset of $A$. The image of $S$ under the function $f$ is the subset of $B$ that consists of the images of $S$. We denote the image of $S$ by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S (t = f(s))\}.$$ 

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Remark: The notation $f(S)$ for the image of the set $S$ under the function $f$ is potentially ambiguous. Here, $f(S)$ denotes a set, and not the value of the function $f$ for the set $S$.

EXAMPLE 7 Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1,$ and $f(e) = 1$. What is the image of the subset $S = \{b, c, d\}$?

Solution: The image of $S$ under the function $f$ is the set $f(S) = \{1, 4\}$. 

4
One-to-one and Onto Functions

**DEFINITION 5** A function \( f \) is said to be **one-to-one**, or an *injection*, if and only if \( f(a) = f(b) \) implies that \( a = b \) for all \( a \) and \( b \) in the domain of \( f \). A function is said to be **injective** if it is one-to-one.

**EXAMPLE 8** Determine whether the function \( f \) from \( \{a, b, c, d\} \) to \( \{1, 2, 3, 4, 5\} \) with \( f(a) = 4 \), \( f(b) = 5 \), \( f(c) = 1 \), and \( f(d) = 3 \) is one-to-one.

*Solution:* The function \( f \) is one-to-one because \( f \) takes on different values at the four elements of its domain. This is illustrated in Figure 3.

![Figure 3](image)

**EXAMPLE 9** Determine whether the function \( f(x) = x^2 \) from the set of integers to the set of integers is one-to-one.

*Solution:* The function \( f(x) = x^2 \) is not one-to-one because, for instance, \( f(1) = f(-1) = 1 \), but \( 1 \neq -1 \).

**EXAMPLE 10** Determine whether the function \( f(x) = x + 1 \) from the set of real numbers to itself is one-to-one.

*Solution:* The function \( f(x) = x + 1 \) is a one-to-one function. To demonstrate this, note that \( x + 1 \neq y + 1 \) when \( x \neq y \).

**EXAMPLE 11** Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker. In this situation, the function \( f \) that assigns a job to each worker is one-to-one. To see this, note that if \( x \) and \( y \) are two different workers, then \( f(x) \neq f(y) \) because the two workers \( x \) and \( y \) must be assigned different jobs.

**DEFINITION 6** A function \( f \) whose domain and codomain are subsets of the set of real numbers is called **increasing** if \( f(x) \leq f(y) \), and **strictly increasing** if \( f(x) < f(y) \), whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \). Similarly, \( f \) is called **decreasing** if \( f(x) \geq f(y) \), and **strictly decreasing** if \( f(x) > f(y) \), whenever \( x < y \) and \( x \) and \( y \) are in the domain of \( f \). (The word strictly in this definition indicates a strict inequality.)

**Remark:** A function that is either strictly increasing or strictly decreasing is guaranteed to be one-to-one.
**Remark:**
A function $f$ is increasing if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$.
A function $f$ is strictly increasing if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$.
A function $f$ is decreasing if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$.
A function $f$ is strictly decreasing if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$.

**DEFINITION 7**
A function $f$ from $A$ to $B$ is called **onto**, or **surjection**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function $f$ is called **surjective** if it is onto.

**FIGURE 5 (a)** One-to-one, not onto

**FIGURE 5 (b)** Onto, not one-to-one

**FIGURE 5 (c)** One-to-one and onto

**FIGURE 5 (d)** Neither one-to-one nor onto

**FIGURE 5 (e)** Not a function
EXAMPLE 12  Let $f$ be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3, f(b) = 2, f(c) = 1,$ and $f(d) = 3$. Is $f$ an onto function?

**Solution:** Because all three elements of the codomain are images of elements in the domain, we see that $f$ is onto. This is illustrated in the figure below. Note that if the codomain were $\{1, 2, 3, 4\}$, then $f$ would not be onto.

![Diagram of an onto function](image)

EXAMPLE 13  Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

**Solution:** The function $f$ is not onto because there is no integer $x$ with $x^2 = -1$.

EXAMPLE 14  Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

**Solution:** The function is onto, because for every integer $y$ there is an integer $x$ such that $f(x) = y$. To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$.

EXAMPLE 15  Consider the function $f$ in Example 11 that assigns jobs to workers. The function $f$ is onto if for every job there is a worker assigned to this job. The function $f$ is not onto when there is at least one job that has no worker assigned to it.

DEFINITION 8  The function $f$ is a **one-to-one correspondence**, or a **bijection**, if it is both one-to-one and onto.
EXAMPLE 16

Let $f$ be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is $f$ a bijection?

Solution: The function $f$ is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, $f$ is a bijection.

We see a graphical representation of $f$ in the diagram above. A function is one-to-one and onto if it can be inverted. The function can be inverted if all of the directed edges can be reversed and all of the elements in the codomain are mapped by the reversed directed edges to individual elements in the domain.

EXAMPLE 17

Let $A$ be a set, the identity function on $A$ is the function $i_A: A \rightarrow A$, where $i_A(x) = x$ for all $x \in A$. In other words, the identity function $i_A$ is the function that assigns each element to itself. The function $i_A$ is one-to-one and onto, so it is a bijection.
Suppose that \( f: A \rightarrow B \).

**To show that \( f \) is injective (one-to-one)** Show that if \( f(x) = f(y) \) for arbitrary \( x, y \in A \) with \( x \neq y \), then \( x = y \).

**To show that \( f \) is not injective** Find particular elements \( x, y \in A \) such that \( x \neq y \) and \( f(x) = f(y) \).

**To show that \( f \) is surjective (onto)** Consider an arbitrary element \( y \in B \) and find an element \( x \in A \) such that \( f(x) = y \).

**To show that \( f \) is not surjective** Find a particular \( y \in B \) such that \( f(x) \neq y \) for all \( x \in A \).

---

**Inverse Functions and Compositions of Functions**

**DEFINITION 9** Let \( f \) be a one-to-one correspondence from the set \( A \) to the set \( B \). The **inverse function** of \( f \) is the function that assigns to an element \( b \) belonging to \( B \) the unique element \( a \) in \( A \) such that \( f(a) = b \). The inverse function is denoted by \( f^{-1} \). Hence, \( f^{-1}(b) = a \) when \( f(a) = b \).

**FIGURE 6** The Function \( f^{-1} \) Is the Inverse of Function \( f \).

**EXAMPLE 18** Let \( f \) be the function for \( \{a, b, c\} \) to \( \{1,2,3\} \) such that \( f(a) = 2 \), \( f(b) = 3 \), and \( f(c) = 1 \). Is \( f \) invertible, and if it is, what is its inverse?

**Solution:** The function \( f \) is invertible because it is a one-to-one correspondence. The inverse function \( f^{-1} \) reverses the correspondence so \( f^{-1}(1) = c \), \( f^{-1}(2) = a \), and \( f^{-1}(3) = b \).

**EXAMPLE 19** Let \( f: \mathbb{Z} \rightarrow \mathbb{Z} \) be such that \( f(x) = x + 1 \). Is \( f \) invertible, and if it is, what is the inverse?

**Solution:** The function \( f \) has an inverse because it is a one-to-one correspondence, as has been shown previously. To reverse the correspondence, suppose that \( y \) is the image of \( x \), so that \( y = x + 1 \). Then \( x = y - 1 \). This means that \( y - 1 \) is the unique element of \( \mathbb{Z} \) that is sent to \( y \) by \( f \). Consequently, \( f^{-1}(y) = y - 1 \).
EXAMPLE 20

Let $f$ be the function from $\mathbb{R}$ to $\mathbb{R}$ with $f(x) = x^2$. Is $f$ invertible?

**Solution:** Because $f(2) = f(-2) = 4$, $f$ is not one-to-one. If an inverse function were defined it would have to assign two elements to 4. Hence, $f$ is not invertible.

EXAMPLE 21

Show that if we restrict the function $f(x) = x^2$ to the set of nonnegative real numbers. Is $f$ in example 20 to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then $f$ is invertible.

**Solution:**

**One-to-one:** The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = 0$. $x^2 - y^2$ can be factored to $(x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Because both $x$ and $y$ are nonnegative, we must have $x = y$. So, this function is one-to-one.

**Onto:** Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if $y$ is a nonnegative real number, there exists a nonnegative real number $x$ such that $x = \sqrt{y}$, which means that $x^2 = y$.

**Invertible:** Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$.

DEFINITION 10

Let $g$ be a function from the set $A$ to the set $B$ and let $f$ be a function from the set $B$ to the set $C$. The **composition** of the functions $f$ and $g$, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$
EXAMPLE 22  Let $g$ be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b, g(b) = c$, and $g(c) = a$. Let $f$ be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3, f(b) = 2$, and $f(c) = 1$. What is the composition of $f$ and $g$, and what is the composition of $g$ and $f$?

Solution: The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$

EXAMPLE 23  Let $f$ and $g$ be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of $f$ and $g$? What is the composition of $g$ and $f$?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

and

$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

Remark: Note that even though $f \circ g$ and $g \circ f$ are defined for functions $f$ and $g$ in Example 21, $f \circ g$ and $g \circ f$ are not equal. In other words, the commutative law does not hold for the composition of functions.
Inverse Functions: When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that \( f \) is a one-to-one correspondence from the set \( A \) to the set \( B \). Then the inverse function \( f^{-1} \) exists and is a one-to-one correspondence from \( B \) to \( A \). The inverse function reverses the correspondence of the original function, so \( f^{-1}(b) = a \) when \( f(a) = b \), and \( f^{-1}(b) = a \) when \( f^{-1}(b) = a \). Hence,

\[
(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a,
\]
and

\[
(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.
\]

The Graphs of Functions

**DEFINITION 11** Let \( f \) be a function from the set \( A \) to the set \( B \). The **graph** of the function \( f \) is the set of ordered pairs \( \{(a, b) | a \in A \text{ and } f(a) = b\} \).

**EXAMPLE 24** Display the graph of the function \( f(n) = 2n + 1 \) from the set of integers to the set of integers.

**Solution:** The graph of \( f \) is the set of ordered pairs of the form \( (n, 2n + 1) \), where \( n \) is an integer. The graph is displayed in Figure 8.

![Figure 8](image.png)

**EXAMPLE 25** Display the graph of the function \( f(x) = x^2 \) from the set of integers to the set of integers.

**Solution:** The graph of \( f \) is the set of ordered pairs of the form \( (x, f(x)) = (x, x^2) \), where \( x \) is an integer. The graph is displayed in Figure 9.
FIGURE 9 The Graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$.

Some Important Functions

**DEFINITION 12** The *floor function* assigns to the real number $x$ the largest integer that is less than or equal to $x$. The value of the floor function at $x$ is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number $x$ the smallest integer that is greater than or equal to $x$. The value of the ceiling function at $x$ is denoted by $\lceil x \rceil$. 
FIGURE 10 (a) Graph of the Floor Function
EXAMPLE 26 These are some values of the floor and ceiling functions:
\[
\left\lfloor \frac{1}{2} \right\rfloor = 0, \left\lceil \frac{1}{2} \right\rceil = 1, \left\lfloor -\frac{1}{2} \right\rfloor = -1, \left\lceil -\frac{1}{2} \right\rceil = 0, \left\lfloor 3.1 \right\rfloor = 3, \left\lceil 3.1 \right\rceil = 4,
\]
\[
\left\lfloor 7 \right\rfloor = 7, \left\lceil 7 \right\rceil = 7
\]
EXAMPLE 27  Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution: To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently, \([100/8] = [12.5] = 13\) bytes are required.
1. Why is $f$ not a function from $\mathbb{R}$ to $\mathbb{R}$?

**Solution:**

<table>
<thead>
<tr>
<th>Part</th>
<th>Function</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>$f(x) = 1/x$</td>
<td>The expression $1/x$ is meaningless for $x = 0$, which is one of the elements in the domain; thus the “rule” is no rule at all. In other words, $f(0)$ is not defined.</td>
</tr>
<tr>
<td>b)</td>
<td>$f(x) = \sqrt{x}$</td>
<td>The function $f(x)$ does not map values to the real numbers when $x &lt; 0$. In other words, when $x &lt; 0$, $f(x)$ is undefined.</td>
</tr>
<tr>
<td>c)</td>
<td>$f(x) = \pm \sqrt{x^2 + 1}$</td>
<td>The “rule” for $f$ is ambiguous. We must have $f(x)$ defined uniquely, but here there are two values associated with every $x$, the positive square root and the negative square root of $x^2 + 1$.</td>
</tr>
</tbody>
</table>

7. Find the domain and range of these functions.

**Solution:**

<table>
<thead>
<tr>
<th>Part</th>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>the function that assigns to each pair of positive integers the maximum of these two integers.</td>
<td>$\mathbb{Z}^+ \times \mathbb{Z}^+$</td>
<td>$\mathbb{Z}^+$</td>
</tr>
<tr>
<td>b)</td>
<td>the function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer</td>
<td>$\mathbb{Z}^+$</td>
<td>${x \in \mathbb{N}</td>
</tr>
<tr>
<td>c)</td>
<td>the function that assigns to a bit string the number of times the block 11 appears</td>
<td>$(0</td>
<td>1)^*$</td>
</tr>
<tr>
<td>d)</td>
<td>the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s.</td>
<td>$(0</td>
<td>1)^*$</td>
</tr>
</tbody>
</table>
15. Determine whether \( f : \mathbb{Z} \times \mathbb{Z} \) is onto.

**Solution:** Recall the definition of onto.

**Definition 7** A function \( f \) from \( A \) to \( B \) is called onto, or surjective, if and only if for every element \( b \in B \) there is an element \( a \in A \) with \( f(a) = b \). A function \( f \) is called a surjection if it is onto.

<table>
<thead>
<tr>
<th>Part</th>
<th>Function</th>
<th>Onto</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>( f(m,n) = m + n )</td>
<td>Yes</td>
<td>Given any integer ( n ), we have ( f(0,n) = n ), so the function is onto.</td>
</tr>
<tr>
<td>b)</td>
<td>( f(m,n) = m^2 + n^2 )</td>
<td>No</td>
<td>Clearly the range contains no negative integers, so the function is not onto.</td>
</tr>
<tr>
<td>c)</td>
<td>( f(m,n) = m )</td>
<td>Yes</td>
<td>Given any integer ( m ), we have ( f(m,25) = m ), so the function is onto. (We could have used any constant in place of 25 in this argument.)</td>
</tr>
<tr>
<td>d)</td>
<td>( f(m,n) =</td>
<td>n</td>
<td>)</td>
</tr>
<tr>
<td>e)</td>
<td>( f(m,n) = m - n )</td>
<td>Yes</td>
<td>Given any integer ( m ), we have ( f(m,0) = m ), so the function is onto.</td>
</tr>
</tbody>
</table>

23. Determine whether each of these functions is a bijection from \( \mathbb{R} \) to \( \mathbb{R} \).

**Solution:**

If we can find an inverse, the function is a bijection. Otherwise we must explain why the function is not one-to-one or not onto.

<table>
<thead>
<tr>
<th>Part</th>
<th>Function</th>
<th>Bijection</th>
<th>Inverse or Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>( f(x) = 2x + 1 )</td>
<td>Yes</td>
<td>One way to determine whether a function is a bijection, is to try to construct its inverse. Solve ( y = 2x + 1 ) for ( x ), obtaining ( g(y) = \frac{y-1}{2} ). Alternatively, we can argue directly. To show that the function is one-to-one, not that if ( 2x + 1 = 2x' + 1 ), then ( x = x' ). To show that the function is onto, not that ( 2 \left( \frac{y-1}{2} \right) + 1 = y ), so every number is in the range.</td>
</tr>
<tr>
<td>b)</td>
<td>( f(x) = x^2 + 1 )</td>
<td>No</td>
<td>Not one-to-one: ( f(1) = f(-1) = 2 ) Not onto, since ( f(x) \geq 1 ).</td>
</tr>
<tr>
<td>c)</td>
<td>( f(x) = x^3 )</td>
<td>Yes</td>
<td>Yes, this function is a bijection, since it has an inverse function ( f(y) = y^{1/3} ) (obtained by solving ( y = x^3 ) for ( x )).</td>
</tr>
<tr>
<td>d)</td>
<td>( f(x) = \frac{(x^2 + 1)}{(x^2 + 2)} )</td>
<td>No</td>
<td>Not one-to-one: ( f(1) = f(-1) = \frac{2}{3} ) Not onto because the range is the set ( {y \in \mathbb{R}</td>
</tr>
</tbody>
</table>