AVL (Adelson, Velski, and Landis) Trees
AVL Trees are binary search trees with a balance condition.
Balance condition: For every node in the tree, the height of the left and right subtrees can differ by at most one.

\[\begin{align*}
\text{{Figure 1. Binary tree satisfying the AVL balance condition}}
\end{align*}\]

\[\begin{align*}
\text{{Figure 2. Binary tree which does not satisfy the AVL balance condition}}
\end{align*}\]

Single Rotation
1. \(k_1 < k_2\)
2. \(\{x_i \in X \mid x_i < k_1\}\)
3. \(\{y_i \in Y \mid k_1 < y_i < k_2\}\)
4. \(\{z_i \in Z \mid z_i > k_2\}\)

\[\begin{align*}
\text{{Figure 3. Clockwise rotation from top to bottom. Counter clockwise rotation from bottom to top}}
\end{align*}\]

\[\begin{align*}
\text{{Figure 4. Recognizing a candidate for single rotation (straight legs)}}
\end{align*}\]
**Single Left Rotation** example: Insert abigail, beulah, and cosette (figure 5), then balance the AVL tree (figure 6) using a single left rotation.

```
    k1
   (0) abigail (2) unbalanced
      k2
     (0) beulah (1)
         Z
        k3
           cosette
```

**Figure 5.** Unbalanced AVL tree

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**Single Right Rotation** example: Insert michelle, ilse, and deborah (figure 7), then balance the AVL tree (figure 8) using a single right rotation.

```
    k1
   (2) michelle (0)
      k2
     (1) ilse (0)
        X
       (0) deborah (0)
```

**Figure 7.** Unbalanced AVL tree.

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The unbalanced AVL tree in figure 9 is balanced using two separate rotations. First a single left rotation is applied resulting in the tree diagramed in figure 10. Then, the tree is rotated again using a single right rotation resulting in the tree in figure 11. Together the rotations are called a left-right double rotation.

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**Left-Right Double Rotation**

1. $k_1 < k_2 < k_3$
2. $\{a_i \in A \mid a_i < k_1\}$
3. $\{b_i \in B \mid k_1 < b_i < k_2\}$
4. $\{c_i \in C \mid k_2 < c_i < k_3\}$
5. $\{d_i \in D \mid d_i > k_3\}$
Right-Left Double Rotation

The unbalanced AVL tree in figure 12 is balanced using two separate rotations. First a single right rotation is applied resulting in the tree diagramed in figure 13. Then, the tree is rotated again using a single left rotation resulting in the tree in figure 14. Together the rotations are called a right-left double rotation.

A candidate for a double rotation is recognized by the dogleg where the AVL tree is unbalanced. The diagram in figure 15 shows an AVL tree in need of a left-right double rotation and the diagram in figure 16 shows an AVL tree in need of a right-left double rotation.

Left-right double rotation example. The diagram in figure 17 shows an AVL tree in need of a left-right double rotation. The diagram in figure 18 shows the resulting balanced AVL tree.

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**Figure 12.** Unbalanced AVL tree in need of a right-left double rotation

**Figure 13.** AVL Tree after a single right rotation

**Figure 14.** Balanced AVL Tree after a right-left double rotation

**Figure 15.** An AVL tree in need of a left-right double rotation

**Figure 16.** An AVL tree in need of a right-left double rotation

**Figure 17.** An AVL tree needing a left-right double rotation

**Figure 18.** An AVL tree after it has been balanced by a left-right double rotation.